# Financial Derivatives FINE 448 <br> 1. Introduction 

Daniel Andrei

## (4) McGill

Winter 2021

## My Background

- MScF 2006, PhD 2012. Lausanne, Switzerland
- 2012-2018: assistant professor of finance at UCLA Anderson
- Since July 2018: assistant professor of finance at McGill Desautels
- I conduct research in the area of theoretical asset pricing, with a special focus on the role of information in financial markets
- More information about me at danielandrei.info


## Outline

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## Useful Information

- Class materials:
- Slides, problem sets, old midterms \& exams: myCourses
- Textbooks:

1. Robert McDonald, Derivatives Markets, Pearson Addison Wesley, third edition, 2009
2. John Hull, Options, Futures, and Other Derivatives, Prentice Hall > eight edition, 2012

- My contact information:
- Office: Bronf 549
- Office hours: TBD
- Phone: (514) 398-5365
- Email: daniel.andrei@mcgill.ca


## Schedule: Mon \& Wed, 8:35am-9:55am (Section 1) and 4:05pm-5:25pm (Section 2), online (recorded)

| Class \# | Day | Class / Other activities |
| :---: | :--- | :--- |
| 1 | Jan. 11, Mon | Class |
| 2 | Jan. 13, Wed | Class |
| 3 | Jan. 18, Mon | Class |
| 4 | Jan. 20, Wed | Class |
| 5 | Jan. 25, Mon | Class |
| 6 | Jan. 27, Wed | Class |
| 7 | Feb. 1, Mon | Class |
| 8 | Feb. 3, Wed | Class |
| 9 | Feb. 8, Mon | Class |
| 10 | Feb. 10, Wed | Class |
| 11 | Feb. 15, Mon | Class |
| 12 | Feb. 17, Wed | Class |
| 13 | Feb. 22, Mon | Class |
| $14 * * *$ | Feb. 24, Wed | MIDTERM (details TBD) |

*** $=$ important dates

| Class \# | Day | Class / Other activities |
| :---: | :--- | :--- |
| - | Mar. 1, Mon | No class (Reading week) |
| - | Mar. 3, Wed | No class (Reading week) |
| 15 | Mar. 8, Mon | Class |
| 16 | Mar. 10, Wed | Class |
| 17 | Mar. 15, Mon | Class |
| 18 | Mar. 17, Wed | Class |
| 19 | Mar. 22, Mon | Class |
| 20 | Mar. 24, Wed | Class |
| 21 | Mar. 29, Mon | Class |
| 22 | Mar. 31, Wed | Class |
| - | Apr. 5, Mon | No class (Easter) |
| 23 | Apr. 7, Wed | Class |
| 24 | Apr. 12, Mon | Class |
| 25 | Apr. 14, Wed | Class |
| 26 | Apr. 15, Thu | Class (makeup) |

- Makeup class: Apr. 15, Thu.
- Midterm: Feb. 24, Wed.
- Final Exam: to be scheduled between Apr. 19 and Apr. 30


## Evaluation

- 5 problem sets (not graded)
- You do not have to submit these ones, but I strongly advise you to practice them in order to be ready for the midterm and the final exam.
- Grade formula:

$$
\begin{aligned}
\text { FINAL GRADE }= & 60 \% \times \text { final exam } \\
& +40 \% \times \text { midterm }
\end{aligned}
$$

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## What is a Derivative?

- Definition
- An agreement between two parties which has a value determined by the price of something else:
- A stock like Apple
- A bond such as a T-Bond
- A currency such as the EUR/CHF rate
- An index such as the S\&P500
- A metal like Gold
- A commodity like Soy beans
- Types
- Options, futures, and swaps.
- Uses
- Risk management
- Speculation
- Reduce transaction costs
- Regulatory arbitrage


## Exchange Traded Derivatives: Contracts Outstanding


source: Bank for International Settlements, Quarterly Review, June 2013.
http://www.bis.org/statistics/extderiv.htm

## Exchange Traded Derivatives: Notional Amount


source: Bank for International Settlements, Quarterly Review, June 2012.
http://www.bis.org/statistics/extderiv.htm

## Over-the-Counter (OTC) Derivatives


source: Bank for International Settlements, Quarterly Review, June 2012.
http://www.bis.org/statistics/derstats.htm

## Buying and Selling a Financial Asset

- Brokers: commissions
- Market-makers: bid-ask spread (reflects the perspective of the market-maker)

| The price at which <br> you can buy | ask (offer) | What the market- <br> maker will sell for |
| :--- | :---: | :--- |
| The price at which <br> you can sell | bid | What the market- <br> maker pays |

- Example: Buy and sell 100 shares of XYZ
- XYZ: bid=\$49.75, ask=\$50, commission=\$15
- Buy: $(100 \times \$ 50)+\$ 15=\$ 5,015$
- Sell: $(100 \times \$ 49.75)-\$ 15=\$ 4,960$
- Transaction cost: $\$ 5,015-\$ 4960=\$ 55$


## Problem 1: $A B C$ stock has a bid price of $\$ 40.95$ and an

 ask price of $\$ 41.05$. Assume that the brokerage fee is quoted as $0.3 \%$ of the bid or ask price.a. What amount will you pay to buy 100 shares?
b. What amount will you receive for selling 100 shares?
c. Suppose you buy 100 shares, then immediately sell 100 shares. What is your round-trip transaction cost?

Problem 1: $A B C$ stock has a bid price of $\$ 40.95$ and an ask price of $\$ 41.05$. Assume that the brokerage fee is quoted as $0.3 \%$ of the bid or ask price.
a. What amount will you pay to buy 100 shares?

$$
(\$ 41.05 \times 100)+(\$ 41.05 \times 100) \times 0.003=\$ 4,117.32
$$

b. What amount will you receive for selling 100 shares?

$$
(\$ 40.95 \times 100)-(\$ 40.95 \times 100) \times 0.003=\$ 4,082.72
$$

c. Suppose you buy 100 shares, then immediately sell 100 shares. What is your round-trip transaction cost?

$$
\$ 4,117.32-\$ 4,082.72=\$ 34.6
$$

## Short-Selling

- When price of an asset is expected to fall
- First: borrow and sell the asset (get \$\$)
- Then: buy back and return the asset (pay \$)
- If price fell in the mean time: Profit $\$=\$ \$-\$$
- The lender must be compensated for dividends received
- Example: Cash flows associated with short-selling a share of IBM for 90 days. Note that the short-seller must pay the dividend, $D$, to the share-lender.

|  | Day 0 | Dividend Ex-Day | Day 90 |
| :--- | :--- | :--- | :--- |
| Action | Borrow shares | - | Return shares |
|  | Sell shares | - | Purchase shares |
| Cash | $+S_{0}$ | $-D$ | $-S_{90}$ |

## Short selling form the perspective of a broker

- A trader places a short sale order
- The broker searches its own inventory, another trader's margin account, or even another brokerage firm's inventory to locate the shares that the client wants to borrow
- If the stock is located, the short sale order is filled and the trader sells the shares in the market
- Once the transaction is placed, the broker does the lending $\Rightarrow$ any benefit (interest for lending out the shares) belongs to the broker
- The broker is responsible for returning the shares (not a big risk due to margin requirements)


## VW's 348\% Two-Day Gain Is Pain for Hedge Funds

From the Wall Street Journal, 2008:

In short squeezes, investors who borrowed and sold stock expecting its value to fall exit from the trades by buying those shares, or covering their positions. That can send a stock upward if shares are hard to come by. When shares are scarce, that can push a company-s market capitalization well beyond a reasonable valuation. [...] Indeed, the recent stock gains left Volkswagen's market value at about $\$ 346$ billion, just below that of the world's largest publicly traded corporation, Exxon Mobil Corp.

## SHARE PRICE IN GERMANY

## Volkswagen

Tuesday: $€ 945$ ( $\$ 1,200$ ), up 82\% October change: up 244\%


## Source: Thomson Reuters

Problem 2: Suppose you short-sell 300 shares of $X Y Z$ stock at $\$ 30.19$ with a commission charge of $0.5 \%$.
Supposing you pay commission charges for purchasing the security to cover the short-sale, how much profit have you made if you close the short-sale at a price of $\$ 29.87$ ?

Problem 2: Suppose you short-sell 300 shares of $X Y Z$ stock at $\$ 30.19$ with a commission charge of $0.5 \%$.
Supposing you pay commission charges for purchasing the security to cover the short-sale, how much profit have you made if you close the short-sale at a price of $\$ 29.87$ ?

Initially, we will receive the proceeds form the sale of the asset, less the proportional commission charge:

$$
300 \times(\$ 30.19)-300 \times(\$ 30.19) \times 0.005=\$ 9,011.72
$$

When we close out the position, we will again incur the commission charge, which is added to the purchasing cost:

$$
300 \times(\$ 29.87)+300 \times(\$ 29.87) \times 0.005=\$ 9,005.81
$$

Finally, we receive total profits of: $\$ 9,011.72-\$ 9,005.81=\$ 5.91$.

## Continuous Compounding

- Terms often used to to refer to interest rates:
- Effective annual rate $r$ : if you invest $\$ 1$ today, $T$ years later you will have

$$
(1+r)^{T}
$$

- Annual rate $r$, compounded $n$ times per year: if you invest $\$ 1$ today, $T$ years later you will have

$$
\left(1+\frac{r}{n}\right)^{n T}
$$

- Annualized continuously compounded rate $r$ : if you invest $\$ 1$ today, $T$ years later you will have

$$
e^{r T} \equiv \lim _{n \rightarrow \infty}\left(1+\frac{r}{n}\right)^{n T}
$$

## Continuous Compounding: Example

- Suppose you have a zero-coupon bond that matures in 5 years. The price today is $\$ 62.092$ for a bond that pays $\$ 100$.
- The effective annual rate of return is

$$
\left(\frac{\$ 100}{\$ 62.092}\right)^{1 / 5}-1=0.10
$$

- The continuously compounded rate of return is

$$
\frac{\ln (\$ 100 / \$ 62.092)}{5}=\frac{0.47655}{5}=0.09531
$$

- The continuously compounded rate of return of $9.53 \%$ corresponds to the effective annual rate of return of $10 \%$. To verify this, observe that

$$
e^{0.09531}=1.10
$$

or

$$
\ln (1.10)=\ln \left(e^{0.09531}\right)=0.09531
$$

## Continous Compounding

- When we multiply exponentials, exponents add. So we have

$$
e^{x} e^{y}=e^{x+y}
$$

This makes calculations of average rate of return easier.

- When using continuous compounding, increases and decreases are symmetric.
- Moreover, continuously compounded returns can be less than - $100 \%$


## Problem 3: Suppose that over 1 year a stock price increases from $\$ 100$ to $\$ 200$. Over the subsequent year it falls back to $\$ 100$.

- What is the effective return over the first year? What is the continuously compounded return?
- What is the effective return over the second year? The continuously compounded return?
- What do you notice when you compare the first- and second-year returns computed arithmetically and continuously?


## Problem 3: Suppose that over 1 year a stock price

 increases from $\$ 100$ to $\$ 200$. Over the subsequent year it falls back to $\$ 100$.- What is the effective return over the first year? What is the continuously compounded return?

$$
\begin{aligned}
\text { effective return } & =\frac{\$ 200-\$ 100}{\$ 100}=100 \% \\
\text { continuously compounded return } & =\ln \left(\frac{\$ 200}{\$ 100}\right)=69.31 \%
\end{aligned}
$$

- What is the effective return over the second year? The continuously compounded return?

$$
\begin{aligned}
\text { effective return } & =\frac{\$ 100-\$ 200}{\$ 200}=-50 \% \\
\text { continuously compounded return } & =\ln \left(\frac{\$ 100}{\$ 200}\right)=-69.31 \%
\end{aligned}
$$

- What do you notice when you compare the first- and second-year returns computed arithmetically and continuously?


## Forward Contracts

- Definition: a binding agreement (obligation) to buy/sell an underlying asset in the future, at a price set today.
- A forward contract specifies:

1. The features and quantity of the asset to be delivered
2. The delivery logistics, such as time, date, and place
3. The price the buyer will pay at the time of delivery

## Bloomberg: CTM <GO>

## <HELP> for explanation, <MENU> for similar functions.



## Bloomberg: SPX <INDEX> CT <GO>



Australia 61297778600 Brazil 551130484500 Europe 442073307500 Germany 496992041210 Hong Kong 85229776000 Japan 81332018900 Singapore 6562121000 U.S. 12123182000 Copuright 2013 Bloomberg Finance L.P. SN 207036 FDT GMT-7:00 G576-1608-2 01-0ct-2013 14:40:46

## Bloomberg: SPZ3 <INDEX > DES < GO >

```
SPZ3 s1689.40 +15.10 matwN N- ---- --x-- Prev 1674.30
At 14:15 d Vol 1117 Op 1679.10 Hi 1692.00 Lo 1675.50 OpenInt 151861
SPZ3 COMB Index 99) Feedback Page 1/2 Futures Contract Description
```


## 1) Contract Information $\quad$ 2) Linked Instruments

SPZ3 Index
S\&P 500 FUTURE Dec13
CME-Chicago Mercantile Exchange

## 3) Notes

S\&P 500 Index Futures
***Effective 11/18/2012, Globex Trading Hours were expanded to MON-FRI: 17:00-16:15 CT with a trading halt from 8:15-15:30 CT***..

| 4) Contracts (CT) | - | Mar:H |  | Jun:M | Sep:U |  | Dec:Z |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Contract Specifications |  | Trading Hours |  |  | 5) Price Chart (GP) |  |  |
| Underlying | SPX Index | - Exchange | - Local |  | - Intraday - History |  | - Curve |
| Contract Size | 250 \$ x index | Electronic | 15:00-14:15 |  |  |  |  |
| Value of 1.0 pt | \$ 250 | Pit |  | 06:30-13:15 |  |  |  |
| Tick Size | 0.10 |  |  |  |  |  |  |
| Tick Value | \$ 25 | Q Related Dates (EXS) |  |  |  |  |  |
| Price 1,689 | 40 index points | Cash Settled |  |  |  |  |  |
| Contract Value | \$ 422,350 | First Trade | Fri | Dec 17, 2010 | \%omm |  | Mos |
| Last Time | 14:15:00 | Last Trade | Thu | Dec 19, 2013 | Prc Chg 1D |  | $1 /+0.902 \%$ |
| Exch Symbol | SP | Valuation Date |  | Dec 20, 2013 | Lifetime High |  | 1,726.50 |
| BBGID | BBG001BT60Z6 |  |  |  | Lifetime Low |  | 1,042.30 |
| Daily Price Limit |  | 8) Holidays (CDR CE) |  |  | Margin Requirements |  |  |
| Up Limit | 1,759.00 | 9) Weekly COT Net Futs (COT) |  |  | Initial Secondary | Speculator | Hedger |
| Down Limit | 1,339.00 |  |  |  | 19,250 | 17,500 |
|  |  |  |  |  | 17,500 | 17,500 |

Australia 61297778600 Brazil 551130484500 Europe 442073307500 Germany 496992041210 Hong Kong 85229776000
Japan $81332018900 \quad$ Singapore 6562121000 U.S. 12123182000
Copuright 2013 Bloomberg Finance L.P. SN 207036 PDT

Copuright 2013 Bloomberg Finance L.P.

## Bloomberg: NGX3 <CMDTY> DES <GO>


NGX3 COMB Comdty 99) Feedback Page 1/2 Futures Contract Description

## 1) Contract Information 2) Linked Instruments

NGX3 Comdty
NATURAL GAS FUTR Nov13
NYM-New York Mercantile Exchange
3) Notes

Natural Gas Futures (HH)
Natural gas accounts for almost a quarter of United States energy consumption, and the NYMEX
Division natural gas futures contract is widely used as a national benchmark price. The futures
4) Contracts (CT) Jan:F Feb:G Mar:H Apr:J May:K Jun:M Jul:N Aug:Q Sep:U Oct:V Nov:X Dec:Z

Contract Specifications Trading Hours
Contract Size
Value of 1.0 pt Tick Size
Tick Value
10,000 MMBtu
\$ 10,000 0.001
\$ 10
Price 3.609
Contract Value Last Time
Exch Symbol
BBGID

## Daily Price Limits

Up Limit
Down Limit
2.109

USD/MMBtu \$ 36,090 10/01/13

NG
BBG000RV94K1
5.109 9) Weekly COT Net Futs (COT)

- Local

15:00-14:15
06:00-11:30

- Exchange Electronic Pit

Q Related Dates (EXS)
First Trade Thu Nov 29, 2007
Last Trade Tue Oct 29, 2013
First Notice Wed Oct 30, 2013
First Delivery Fri Nov 1, 2013
Last Delivery Sat Nov 30, 2013
8) Holidays (CDR NM)
5) Price Chart (GP)

O Intraday $\bullet$ History
Australia 61297778600 Brazil 551130484500 Europe 442073307500 Germany 496992041210 Hong Kong 85229776000
Japan $81332018900 \quad$ Singapore 6562121000 U.S. 12123182000
SN 207036 PDT

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## Payoff (Value at Expiration) of a Forward Contract

- Every forward contract has both a buyer and a seller.
- The term long is used to describe the buyer and short is used to describe the seller.
- Payoff for
- Long forward $=$ Spot price at expiration - Forward price
- Short forward = Forward price - Spot price at expiration
- Example: S\&R index:
- Today: Spot price $=\$ 1,000.6$-month forward price $=\$ 1,020$
- In 6 months at contract expiration: Spot price $=\$ 1,050$
- Long position payoff $=\$ 1,050-\$ 1,020=\$ 30$
- Short position payoff $=\$ 1,020-\$ 1,050=-\$ 30$


## Payoff Diagram for a Forward



Problem 4: Suppose you enter in a long 6-month forward position at a forward price of $\$ 50$. What is the payoff in 6 months for prices of $\$ 40, \$ 45, \$ 50, \$ 55$, and $\$ 60$ ?

Problem 4: Suppose you enter in a long 6-month forward position at a forward price of $\$ 50$. What is the payoff in 6 months for prices of $\$ 40, \$ 45, \$ 50, \$ 55$, and $\$ 60$ ?

The payoff to a long forward at expiration is equal to:
Payoff to long forward $=$ Spot price at expiration - Forward price
Therefore, we can construct the following table:

| Price of asset in 6 months | Payoff ot the long forward |
| :---: | :---: |
| 40 | -10 |
| 45 | -5 |
| 50 | 0 |
| 55 | 5 |
| 60 | 10 |

## Alternative ways to buy a stock

## 1. Outright Purchase:

- Pay $S_{0}$
- Receive security


## 2. Forward:

- Pay $F_{0, T}=$ ?
- Receive security

Time T

- A forward contract is an arrangement in which you both pay for the stock and receive it at time $T$, with the time $T$ price specified at time 0 .
- What should you pay for the stock in this case?
- Arbitrage ensures that there is a very close relationship between prices and forward prices


## Pricing a Forward Contract

- Let $S_{0}$ be the spot price of an asset at time 0 , and $r$ the continuously compounded interest rate. Assume that dividends are continuous and paid at a rate $\delta$.
- Then the forward price at a future time $T$ must satisfy

$$
\begin{equation*}
F_{0, T}=S_{0} e^{(r-\delta) T} \tag{1}
\end{equation*}
$$

- Suppose that $F_{0, T}>S_{0} e^{(r-\delta) T}$. Then an investor can execute the following trades at time 0 (buy low and sell high) and obtain an arbitrage profit:

Cash Flows

| Transaction | Time 0 | Time $T$ (expiration) |
| :--- | :--- | :--- |
| Buy tailed position in stock ( $e^{-\delta T}$ units) | $-S_{0} e^{-\delta T}$ | $S_{T}$ |
| Borrow $S_{0} e^{-\delta T}$ | $+S_{0} e^{-\delta T}$ | $-S_{0} e^{(r-\delta) T}$ |
| Short forward | 0 | $F_{0, T}-S_{T}$ |
| Total | 0 | $F_{0, T}-S_{0} e^{(r-\delta) T}>0$ |

## Pricing a Forward Contract (cont'd)

- Suppose that $F_{0, T}<S_{0} e^{(r-\delta) T}$. Then an investor can execute the following trades at time 0 (buy low and sell high) and obtain once again an arbitrage profit:

|  | Cash Flows |  |
| :--- | :--- | :--- |
| Transaction | Time 0 | Time $T$ (expiration) |
| Short tailed position in stock $\left(e^{-\delta T}\right.$ units) | $S_{0} e^{-\delta T}$ | $-S_{T}$ |
| Lend $S_{0} e^{-\delta T}$ | $-S_{0} e^{-\delta T}$ | $S_{0} e^{(r-\delta) T}$ |
| Long forward | 0 | $S_{T}-F_{0, T}$ |
| Total | 0 | $S_{0} e^{(r-\delta) T}-F_{0, T}>0$ |

- Consequently, and assuming that the non-arbitrage condition holds, we have

$$
F_{0, T}=S_{0} e^{(r-\delta) T}
$$

## Forward Contracts vs Futures Contracts

- Forward and futures contracts are essentially the same except for the daily resettlement feature of futures contracts, called marking-to-market.
- Because futures are exchange-traded, they are standardized and have specified delivery dates, locations, and procedures.
- Plenty of information is available from: www.cmegroup.com


## The S\&P 500 Futures Contract

## Specifications for the S\&P 500 index futures contract

- Underlying: S\&P 500 index
- Where traded: Chicago Mercantile Exchange
- Size: $\$ 250 \times$ S\&P 500 index
- Months: Mar, Jun, Sep, Dec
- Trading ends: Business day prior to determination of settlement price
- Settlement: Cash-settled, based upon opening price of S\&P 500 on third Friday of expiration month
- Suppose the futures price is 1100 and you wish to enter into 8 long futures contracts.
- The notional value of 8 contracts is

$$
8 \times \$ 250 \times 1100=\$ 2,000 \times 1100=\$ 2.2 \text { million }
$$

## The S\&P 500 Futures Contract (cont'd)

- Suppose that there is $10 \%$ margin and weekly settlement (in practice settlement is daily). The margin on futures contracts with a notional value of $\$ 2.2$ million is $\$ 220,000$.
- The margin balance today from long position in 8 S\&P 500 futures contracts is

| Week | Multiplier (\$) | Futures Price | Price Change | Margin Balance (\$) |
| :---: | :--- | :--- | ---: | ---: |
| 0 | 2000.00 | 1100.00 | - | $220,000.00$ |

- Over the first week, the futures price drops 72.01 points to 1027.99 . On a mark-to-market basis, we have lost

$$
\$ 2,000 \times(-72.01)=-\$ 144,020
$$

- Thus, if the continuously compounded interest rate is $6 \%$, our margin balance after one week is

$$
\$ 220,000 \times e^{0.06 \times 1 / 52}-\$ 144,020=\$ 76,233.99
$$

## The S\&P 500 Futures Contract (cont'd)

- Because we have a $10 \%$ margin, a $6.5 \%$ decline in the futures price results in a $65 \%$ decline in margin. The margin balance after the first week is

| Week | Multiplier (\$) | Futures Price | Price Change | Margin Balance (\$) |
| :---: | :--- | :--- | ---: | ---: |
| 0 | 2000.00 | 1100.00 | - | $220,000.00$ |
| 1 | 2000.00 | 1027.99 | -72.01 | $76,233.99$ |

- The decline in margin balance means the broker has significantly less protection should we default. For this reason, participants are required to maintain the margin at a minimum level, called the maintenance margin. This is often set at $70 \%$ to $80 \%$ of the initial margin level.
- In this example, the broker would make a margin call, requesting additional margin.
- We can go on for a period of 10 weeks, assuming weekly marking-to-market and a continuously compounded risk-free rate of 6\%.


## The S\&P 500 Futures Contract (cont'd)

- The margin balance after a period of 10 weeks is

| Week | Multiplier (\$) | Futures Price | Price Change | Margin Balance (\$) |
| :---: | :--- | :--- | ---: | ---: |
| 0 | 2000.00 | 1100.00 | - | $220,000.00$ |
| 1 | 2000.00 | 1027.99 | -72.01 | $76,233.99$ |
| 2 | 2000.00 | 1037.88 | 9.89 | $96,102.01$ |
| 3 | 2000.00 | 1073.23 | 35.35 | $166,912.96$ |
| 4 | 2000.00 | 1048.78 | -24.45 | $118,205.66$ |
| 5 | 2000.00 | 1090.32 | 41.54 | $201,422.13$ |
| 6 | 2000.00 | 1106.94 | 16.62 | $234,894.67$ |
| 7 | 2000.00 | 1110.98 | 4.04 | $243,245.86$ |
| 8 | 2000.00 | 1024.74 | -86.24 | $71,046.69$ |
| 9 | 2000.00 | 1007.30 | -17.44 | $36,248.72$ |
| 10 | 2000.00 | 1011.65 | 4.35 | $44,990.57$ |

- The 10 -week profit on the position is obtained by subtracting from the final margin balance the future value of the original margin investment:

$$
\$ 44,990.57-\$ 220,000 \times e^{0.06 \times 10 / 52}=-\$ 177,562.60
$$

## The S\&P 500 Futures Contract (cont'd)

- What if the position had been forwarded rather than a futures position, but with prices the same? In that case, after 10 weeks our profit would have been

$$
(1011.65-1100) \times \$ 2,000=-\$ 176,700
$$

- The futures and forward profits differ because of the interest earned on the mark-to-market proceeds (in the present cases, we have founded losses as they occurred and not at expiration, which explains the loss).


## Uses Of Index Futures

- Why buy an index futures contract instead of synthesizing it using the stocks in the index? - Lower transaction costs
- Asset allocation: switching investments among asset classes. Example: invested in the S\&P 500 index and wish to temporarily invest in bonds instead of the index. What to do?
- Alternative \#1: sell all 500 stocks and invest in bonds
- Alternative \#2: take a short forward position in S\&P 500 index
- General asset allocation: futures overlay, alpha-porting
- Cross-hedging: hedge portfolios that are not exactly the index
- Risk management for stock-pickers
- More in Chapter 5, Section 5.5 of McDonald (2009)


## Call Options

- A non-binding agreement (right but not an obligation) to buy an asset into the future, at a price set today
- Preserves the upside potential, while at the same time eliminating the downside
- The seller of a call option is obligated to deliver if asked


## Definition and terminology

- A call option gives the owner the right but not the obligation to buy the underlying asset at a predetermined price during a predetermined time period
- Strike (or exercise) price: the amount paid by the option buyer for the asset if he/she decides to exercise
- Exercise: the act of paying the strike price to buy the asset
- Expiration: the date by which the option must be exercised or becomes worthless
- Exercise style: specifies when the option can be exercised
- European-style: can be exercised only at expiration date
- American-style: can be exercised at any time before expiration
- Bermudan-style: can be exercised during specified periods


## Moneyness

- In-the-money option: positive payoff if exercised immediately
- At-the-money option: zero payoff if exercised immediately
- Out-of-the-money option: negative payoff if exercised immediately


## Bloomberg: WFC US <EQUITY> OMON <GO>



## Call Option Example

- Consider a call option on the S\&R index with 6 months to expiration and strike price of $\$ 1,000$.
- In six months at contract expiration: if spot price is
- $\$ 1,100 \Rightarrow$ call buyer's payoff $=\$ 1,100-\$ 1,000=\$ 100$, call seller's payoff $=-\$ 100$
- $\$ 900 \Rightarrow$ call buyer's payoff $=\$ 0$, call seller's payoff $=\$ 0$
- The payoff of a call option is then

$$
\begin{equation*}
C_{T}=\max \left[S_{T}-K, 0\right] \tag{2}
\end{equation*}
$$

where $K$ is the strike price, and $S_{T}$ is the spot price at expiration.

- The option profit is computed as

Call profit $=\max \left[S_{T}-K, 0\right]-$ future value of premium

## Diagrams for Purchased Call




## Diagrams for Written Call




Problem 5: Consider a call option on the S\&R index with 6 months to expiration and strike price of $\$ 1,000$. The future value of the option premium is $\$ 95.68$. For the figure below, which plots the profit on a purchased call, find the S\&R index price at which the call option diagram intersects the $x$-axis.


Problem 5: Consider a call option on the S\&R index with 6 months to expiration and strike price of $\$ 1,000$. The future value of the option premium is $\$ 95.68$. For the figure below, which plots the profit on a purchased call, find the S\&R index price at which the call option diagram intersects the $x$-axis.


The profit of the long call option is:

$$
\max \left[0, S_{T}-\$ 1,000\right]-\$ 95.68
$$

To find the S\&R index price at which the call option diagram intersects the $x$-axis, we have to set the above equation equal to zero. We get

$$
S_{T}=\$ 1,095.68
$$

## Bloomberg: WFC 10/19/13 C41 <EQUITY> OV <GO>



## Put Options

- A put option gives the owner the right but not the obligation to sell the underlying asset at a predetermined price during a predetermined time period.
- The payoff of the put option is

$$
\begin{equation*}
P_{T}=\max \left[K-S_{T}, 0\right] \tag{4}
\end{equation*}
$$

- The option profit is computed as

$$
\begin{equation*}
\text { Put profit }=\max \left[K-S_{T}, 0\right]-\text { future value of premium } \tag{5}
\end{equation*}
$$

## Diagrams for Purchased Put




## Diagrams for Written Put




## Problem 6: Suppose the stock price is $\$ 40$ and the

 effective annual interest rate is $8 \%$. Draw payoff and profit diagrams for a 40 -strike put with a premium of $\$ 3.26$ and maturity of 1 year.
## Problem 6: Suppose the stock price is $\$ 40$ and the

 effective annual interest rate is $8 \%$. Draw payoff and profit diagrams for a 40 -strike put with a premium of $\$ 3.26$ and maturity of 1 year.In order to be able to draw the profit diagram, we need to find the future value of the put premium:

$$
\begin{aligned}
F V(\text { premium }) & =\$ 3.26 \times(1+0.08) \\
& =\$ 3.5208
\end{aligned}
$$

We get the following payoff and profit diagram:


## Put-Call Parity

- Suppose you are buying a call option and selling a put option on a non-dividend paying stock. Both options have maturity $T$ and strike price $K$ :




## Put-Call Parity (cont'd)

- Your payoff at maturity is

$$
\begin{aligned}
C_{T}-P_{T} & =\max \left[S_{T}-K, 0\right]-\max \left[K-S_{T}, 0\right] \\
& =\max \left[S_{T}-K, 0\right]+\min \left[S_{T}-K, 0\right] \\
& =S_{T}-K
\end{aligned}
$$

- We have two strategies with the same payoff at maturity:
- Buy a call and sell a put, thus paying a premium of $C_{t}-P_{t}$ today
- Buy a share of the stock and borrow $P V(K)$, thus paying a premium of $S_{t}-P V(K)$ today
- Positions that have the same payoff should have the same cost (Law of one price):

$$
\begin{equation*}
C_{t}-P_{t}=S_{t}-P V(K) \tag{6}
\end{equation*}
$$

- Equation (6) is known as put-call parity, and one of the most important relations in options.


## Put-Call Parity (cont'd)

- Parity provides a cookbook for the synthetic creation of options. It tells us that

$$
\begin{equation*}
C_{t}=P_{t}+S_{t}-P V(K) \tag{7}
\end{equation*}
$$

and that

$$
\begin{equation*}
P_{t}=C_{t}-S_{t}+P V(K) \tag{8}
\end{equation*}
$$

- The first relation says that a call is equivalent to a leveraged position on the underlying asset, which is insured by the purchase of a put. The second relation says that a put is equivalent to a short position on the stock, insured by the purchase of a call
- Parity generally fails for American-style options, which may be exercised prior to maturity.


## Why Does the Price of an At-the-Money call Exceed the Price of an At-the-Money put?

- Parity shows that the reason for the call being more expensive is the time value of money:

$$
\begin{equation*}
C_{t}-P_{t}=K-P V(K)>0 \tag{9}
\end{equation*}
$$

- A common erroneous explanation is that the profit on a call is unlimited, while the profit on a put can be no greater than the strike price, which seems to suggest that the call should be more expensive than the put.
- This argument also seems to suggest that every stock is worth more than its price!


## Problem 7: The S\&R index price is $\$ 1,000$ and the

 effective 6 -month interest rate is $2 \%$. Suppose the premium on a 6 -month S\&R call is $\$ 109.20$ and the premium on a 6 -month put with the same strike price is $\$ 60.18$. What is the strike price?Problem 7: The S\&R index price is $\$ 1,000$ and the effective 6 -month interest rate is $2 \%$. Suppose the premium on a 6 -month S\&R call is $\$ 109.20$ and the premium on a 6 -month put with the same strike price is $\$ 60.18$. What is the strike price?

This question is a direct application of the Put-Call Parity:

$$
\begin{aligned}
C_{t}-P_{t} & =S_{t}-P V(K) \\
\$ 109.20-\$ 60.18 & =\$ 1,000-\frac{K}{1.02} \\
K & =\$ 970.00
\end{aligned}
$$

## Put-Call Parity for Dividend Paying Stocks

- If the stock is paying dividends over the lifetime of the option, the put-call parity becomes

$$
\begin{equation*}
C_{t}-P_{t}=\left[S_{t}-P V(\text { Div })\right]-P V(K) \tag{10}
\end{equation*}
$$

where $P V$ (Div) is the present value of the stream of dividends paid on the stock until maturity.

- Hence, we can write

$$
\begin{align*}
& C_{t}=P_{t}+\left[S_{t}-P V(\text { Div })\right]-P V(K)  \tag{11}\\
& P_{t}=C_{t}-\left[S_{t}-P V(\text { Div })\right]+P V(K) \tag{12}
\end{align*}
$$

- Equations (11)-(12) help us to find maximum and minimum option prices.


## Maximum and Minimum Option Prices: Call Price



Maximum and Minimum Option Prices: Call Price


Maximum and Minimum Option Prices: Call Price


## Maximum and Minimum Option Prices: Call Price



# Maximum and Minimum Option Prices: Put Price 



## Problem 8 (Minimum and Maximum Bounds, Arbitrage)

- A one-month European put option on a non-dividend paying stock is currently selling for 2.50 . The option has a strike of 50 and the underlying is currently worth 46 . The interest rate is $10 \%$.
- Is there an arbitrage opportunity? If yes, show how you would implement it.


## Problem 9 (Building Payoffs)

Below is a payoff diagram for a position. All options have 1 year to maturity and the stock price today is $\$ 100$. The yearly interest rate (continuously compounded) is $8 \%$. The underlying asset (the stock) is not paying any dividends.


| Option | Call(90) | Call(100) | Call(110) |
| :---: | :--- | :--- | :--- |
| Position |  |  |  |

## Example: Equity-Linked CDs

- A 1,999 First Union National Bank CD promises to repay in 5.5 years initial invested amount and $70 \%$ of the gain in S\&P 500 index (this is a principal protected equity-linked $C D$ )
- Assume $\$ 10,000$ invested when S\&P $500=1,300$
- Final payoff is
$\$ 10,000 \times\left(1+0.7 \times \max \left[0, \frac{S_{\text {final }}}{1300}-1\right]\right)$
where $S_{\text {final }}=$ value of the S\&P 500 after 5.5 years.



## Options are Insurance: Insuring a Long Position (Floors)

- A put option is combined with a position in the underlying asset
- Goal: to insure against a fall in the price of the underlying asset (when one has a long position in that asset)




## Options are Insurance: Insuring a Short Position (Caps)

- A call option is combined with a position in the underlying asset
- Goal: to insure against an increase in the price of the underlying asset (when one has a short position in that asset)




## Various Strategies: Payoffs



## Various Strategies: Positions

Bull Spread

|  | $K_{\text {low }}$ | $K_{\text {ATM }}$ | $K_{\text {high }}$ |
| :---: | :---: | :---: | :---: |
| Call |  | Buy | Sell |
| Put |  |  |  |

Collar

|  | $K_{\text {low }}$ | $K_{\text {ATM }}$ | $K_{\text {high }}$ |
| :---: | :---: | :---: | :---: |
| Call |  |  | Sell |
| Put | Buy |  |  |

Strangle

|  | $K_{\text {low }}$ | $K_{A T M}$ | $K_{\text {high }}$ |
| :---: | :---: | :---: | :---: |
| Call |  |  | Buy |
| Put | Buy |  |  |

Butterfly Spread
Butterfly Spread

|  | $K_{\text {low }}$ | $K_{\text {ATM }}$ | $K_{\text {high }}$ |
| :---: | :---: | :---: | :---: |
| Call | Buy | Sell $(2)$ | Buy |
| Put |  |  |  |

Straddle

|  | $K_{\text {low }}$ | $K_{\text {ATM }}$ | $K_{\text {high }}$ |
| :---: | :---: | :---: | :---: |
| Call |  | Buy |  |
| Put |  | Buy |  |


|  | $K_{\text {low }}$ | $K_{\text {ATM }}$ | $K_{\text {high }}$ |
| :--- | :--- | :--- | :--- |
| Call |  | Buy | Sell (n) |
| Put |  |  |  |

Note that you can achieve the same results with different combinations (but always at the same cost!)

## Various Strategies: Rationales

## Bull Spread

- You believe a stock will appreciate $\Rightarrow$ buy a call option (forward position insured)
- You can lower the cost if you are willing to reduce your profit should the stock appreciate $\Rightarrow$ sell a call with higher strike
- Surprisingly, you can achieve the same result by buying a low-strike put and selling a high-strike put
- Opposite: bear spread


## Strangle

- To reduce the premium of a straddle, you can buy out-of-the-money options rather than at-the-money options.
- Opposite: written strangle


## Collar

- A collar is fundamentally a short position (resembling a short forward contract)
- Often used for insurance when we own a stock (collared stock)
- The collared stock looks like a bull spread; however, it arises from a different set of transactions
- Opposite: written collar


## Butterfly Spread

- A butterfly spread is a written straddle to which we add two options to safeguard the position: An out-of-the money put and an out-of-the money call.
- A butterfly spread can be thought of as a written straddle for the timid (or for the prudent!)
- Opposite: long iron butterfly


## Straddle

- A straddle can profit from stock price moves in both directions
- The disadvantage is that it has a high premium because it requires purchasing two options
- Opposite: written straddle


## Ratio Spread

- Ratio spreads involve buying one option and selling a greater quantity ( n ) of an option with a more out-of-the money strike
- The ratio (i.e., " 1 by $n$ ") is the number of short options divided by the number of long options
- The options are either both calls or both puts
- It is possible to construct ratio spreads with zero premium $\Rightarrow$ we can construct insurance that costs nothing if it is not needed!


## Bloomberg: Products $\rightarrow$ Option Strategies



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## Bloomberg: Products $\rightarrow$ Option Strategies



## Lakonishok, Lee, Pearson, and Poteshman, Option Market Activity, The Review of Financial Studies, 2006

- Stylized facts about option trading
- Written option positions are more common than purchased positions
- About 4 times more purchased calls than puts
- Main driver of option market activity is speculating on the direction of underlying stock movements
- Option trading strategies
- Small fraction of volatility trading strategies (straddles and strangles)
- Large fraction of covered-call strategies
- Option market activity during the stock market bubble of the late 1990s
- Call buying and put writing increased dramatically (mostly on growth stocks)
- Purchased puts less common (little appetite for betting against the bubble)


## Collars in Acquisitions: WorldCom/MCI

- On October 1, 1997, WorldCom Inc. CEO (Bernard Ebbers) sent the following note to the CEO of MCl (Bert Roberts), and it was also released through the typical newswires:
- "I am writing to inform you that this morning WorldCom is publicly announcing that it will be commencing an offer to acquire all the outstanding shares of MCl for $\$ 41.50$ of WorldCom common stock per MCl share. The actual number of shares of WorldCom common stock to be exchanged for each MCl share in the exchange offer will be determined by dividing $\$ 41.50$ by the 20-day average of the high and low sales prices for WorldCom common stock prior to the closing of the exchange offer, but will not be less than 1.0375 shares (if WorldCom's average stock price exceeds \$40) or more than 1.2206 shares (if WorldCom's average stock price is less than \$34)."


## Collars in Acquisitions: WorldCom/MCI (cont'd)

- The payoff is contingent upon price of WorldCom's 20-day average stock price prior to the closing exchange offer:

- Problem Set 1 is available


## Financial Derivatives FINE 448

2. Commodity Futures: Basis Risk \& Cross Hedging

Daniel Andrei

## (4) McGill

Winter 2021

## Commodity Forwards and Futures

- Commodities are complex because every commodity market differs in the details. For example:
- Storage is not possible for electricity
- Gold is durable and relatively inexpensive to store (compared to its value)
- Some commodities feature seasonality in production (for example, corn in the United States is harvested primarily in the fall)


## Introduction to Commodity Forwards and Futures

- Financial forward prices are described by the general formula

$$
F_{0, T}=S_{0} e^{(r-\delta) T}
$$

- At a general level, commodity forward prices can be described by the same formula. There are, however, important differences:
- For financial assets $\delta$ is the dividend yield, whereas for commodities $\delta$ is the commodity lease rate
- While the dividend yield for a financial asset can typically be observed directly, the lease rate for a commodity can be estimated only by observing the forward price
- The formula for a commodity forward price is

$$
\begin{equation*}
F_{0, T}=S_{0} e^{\left(r-\delta_{l}\right) T} \tag{1}
\end{equation*}
$$

## Forward Prices and The Lease Rate

- When we observe the forward price, we can infer the lease rate. Specifically, if the forward price is $F_{0, T}$, the annualized lease rate is

$$
\begin{equation*}
\delta_{l}=r-\frac{1}{T} \ln \left(\frac{F_{0, T}}{S_{0}}\right) \tag{2}
\end{equation*}
$$

- If instead we use an effective annual interest rate, the effective annual lease rate is

$$
\begin{equation*}
\delta_{l}=\frac{1+r}{\left(F_{0, T} / S_{0}\right)^{1 / T}}-1 \tag{3}
\end{equation*}
$$

Suppose that on June 6, 2001, the gold spot price is $\$ 265.7$ and the gold future price with maturity in
December is $\$ 269$. The June to December interest rate (annualized, effective) is $3.9917 \%$. What is the annualized 6-month gold lease rate?

- Using equation (3), the annualized 6-month lease rate is

$$
\text { 6-month lease rate }=\frac{1+0.039917}{(269 / 265.7)^{1 / 0.5}}-1=1.456 \%
$$

## The Cost of Carry

$$
\begin{equation*}
F_{0, T}=S_{0} e^{\left(r-\delta_{l}\right) T} \tag{4}
\end{equation*}
$$

- Commodities can be subject to significant storage costs
- Ownership of the physical commodity provides benefits that are not obtained by holders of futures contracts
- For an oil refiner, the crude oil in inventory is an essential input
- For a food producer, the corn in inventory is an essential input
- One can thus write

$$
\begin{equation*}
\delta_{l}=\text { storage cost }+ \text { convenience yield }=-u+y \tag{5}
\end{equation*}
$$

- The futures price is then

$$
\begin{equation*}
F_{0, T}=S_{0} e^{(r+u-y) T}=S_{0} e^{(c-y) T} \tag{6}
\end{equation*}
$$

where $c=r+u$ is the cost of carry.

## Forward Curves for Various Commodities, May 5, 2004

Corn Futures Price (cents per bushel)


Crude Oil Futures Price (\$ per barrel)


Gasoline Futures Price (cents per gallon)


Gold Futures Price (\$ per ounce)


## Cross Hedging

- Cross hedging $=$ the use of a derivative on one asset to hedge another asset.
- Example: Jet fuel futures do not exist (or they are very illiquid) in United States, but firms sometimes hedge jet fuel with crude oil futures, or heating oil futures.
- If we own a quantity of jet fuel and hedge by holding $N$ crude oil futures contracts, our mark-to-market profit depends on the change in the jet fuel price and the change in the futures price:

$$
\begin{equation*}
\left(S_{t}-S_{t-1}\right)+N\left(F_{t}-F_{t-1}\right) \tag{7}
\end{equation*}
$$

where $S_{t}$ is the price of jet fuel and $F_{t}$ the heating oil futures price.

- What is the optimal number of futures contracts for hedging, $N^{*}$ ?


## Cross Hedging Example

- An airline expects to purchase 2 million gallons of jet fuel in 1 month and decides to use heating oil futures for hedging.
- Define:
- $\Delta S=$ change in the jet fuel price per gallon
- $\Delta F=$ change in the heating oil futures price price per gallon
- Standard deviations and correlations are (data on next slide)

$$
\begin{equation*}
\sigma_{F}=0.0313, \quad \sigma_{S}=0.0263, \quad \text { and } \rho=0.9284 \tag{8}
\end{equation*}
$$

- Consider the linear relationship:

$$
\begin{equation*}
\Delta S=a+h^{*} \times \Delta F+\epsilon \tag{9}
\end{equation*}
$$

- $h^{*}=\rho \frac{\sigma_{S}}{\sigma_{F}}$ is the minimum variance hedge ratio
- $R^{2}=\rho^{2}$ is the hedge effectiveness
- In this example, $h^{*}=0.7777$ and $R^{2}=0.8619$
(Without basis risk, these numbers would be $h^{*}=1$ and $R^{2}=1$.)


## Cross Hedging Example



## Cross Hedging Example

- To calculate the number of contracts that should be used in hedging, define:
- $Q_{A}$ : Size of position being hedged (units)
- $Q_{F}$ : Size of one futures contract (units)
- $N^{*}$ : Optimal number of futures contracts for hedging
- The futures contracts should be $h^{*} Q_{A}$ units of the asset. The number of futures contracts required is therefore given by

$$
\begin{equation*}
N^{*}=\frac{h^{*} Q_{A}}{Q_{F}} \tag{10}
\end{equation*}
$$

- Each heating oil contract traded on NYMEX is on 42,000 gallons of heating oil. Thus, the optimal number of contracts is

$$
\begin{equation*}
N^{*}=\frac{0.7777 \times 2,000,000}{42,000}=37.03 \tag{11}
\end{equation*}
$$

or, rounding to the nearest whole number, 37 .

## Cross Hedging

- When futures are used for hedging, a small adjustment, known as tailing the hedge, can be made to allow for the impact of daily settlement.
- This does not change the intuition, but only the result (slightly).
- If you want to learn more about forwards and futures, read chapter 5, "Determination of forward and futures prices" in Hull (eight edition) and chapters 5, "Financial forwards and futures" and 6, "Commodity Forwards and Futures" in McDonald (third edition)


# Financial Derivatives FINE 448 3. Binomial Option Pricing 

Daniel Andrei

## (4) McGill

Winter 2021

## Outline

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II Call Options ..... 6

- A One-Period Binomial Tree ..... 7
- The Binomial Solution ..... 15
- A Two-Period Binomial Tree ..... 22
- Many Binomial Periods ..... 27
III Put Options ..... 28
IV Path-Dependent Options ..... 34
V Uncertainty in the Binomial Model ..... 39
VI Risk-Neutral Pricing ..... 45
VII American Options ..... 53
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## Outline

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V/II American Options ..... 53
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## Discrete-Time Option Pricing: The Binomial Model

- Until now, we have looked only at some basic principles of option pricing
- We examined payoff and profit diagrams, and upper/lower bounds on option prices
- We saw that with put-call parity we could price a put or a call based on the prices of the combinations on instruments that make up the synthetic version of the put or call.
- What we need to be able to do is price a put or a call without the other instrument.
- In this section, we introduce a simple means of pricing an option.


## Discrete-Time Option Pricing: The Binomial Model (cont'd)

- The approach we take here is called the binomial tree.
- The word "binomial" refers to the fact that there are only two outcomes (we let the underlying price move to only one of two possible new prices).
- It may appear that this framework oversimplifies things, but the model can eventually be extended to encompass all possible prices.


## Outline

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## The Binomial Option Pricing Model

- The binomial option pricing model assumes that, over a period of time, the price of the underlying asset can move only up or down by a specified amount-that is, the asset price follows a binomial distribution:



## A Simple Example

- XYZ does not pay dividends and its current price is $\$ 41$. In one year the price can be either $\$ 59.954$ or $\$ 32.903$, i.e., $u=1.4623$ and $d=0.8025$.
- Consider a European call option on the stock of XYZ, with a $\$ 40$ strike an 1 year to expiration. The continuously compounded risk-free interest rate is $8 \%$.
- We wish to determine the option price:

$$
\begin{aligned}
& u S=59.954 \\
& C_{u}=\max [0,59.954-40]=19.954 \\
& C=? \\
& d S=32.903 \\
& C_{d}=\max [0,32.903-40]=0
\end{aligned}
$$

## A Simple Example (cont'd)

- Let us try to find a portfolio that mimics the option (replicating portfolio).
- We have two instruments: shares of stock and a position in bonds (i.e., borrowing or lending).
- To be specific, we wish to find a portfolio consisting of $\Delta$ shares of stock and a dollar amount $B$ in borrowing or lending, such that the portfolio imitates the option whether the stock rises or falls.
- The value of this replicating portfolio at maturity is:

$$
\begin{cases}59.954 \Delta+e^{0.08} B & =19.954  \tag{1}\\ 32.903 \Delta+e^{0.08} B & =0\end{cases}
$$

## A Simple Example (cont'd)

- The unique solution of this system of 2 equations with 2 unknowns is

$$
\begin{equation*}
\Delta=0.738, B=-22.405 \tag{2}
\end{equation*}
$$

i.e., buy 0.738 shares of XYZ and borrow $\$ 22.405$ at the risk-free rate.

- In computing the payoff for the replicating portfolio, we assume that we sell the shares at the market price and that we repay the borrowed amount, plus interest.
- Thus, we obtain that the option and the replicating portfolio have the same payoff: $\$ 19.954$ if the stock price goes up and $\$ 0$ if the stock price goes down.


## A Simple Example (cont'd)

- By the law of one price, positions that have the same payoff should have the same cost.
- The price of the option must be

$$
\begin{equation*}
C=\underbrace{0.738 \times \$ 41}_{\text {risky }}-\underbrace{\$ 22.405}_{\text {risk-free }}=\$ 7.839 \tag{3}
\end{equation*}
$$

## Arbitraging a Mispriced Option

- Suppose that the market price for the option is $\$ 8$ instead of $\$ 7.839$ (the option is overpriced).
- We can sell the option and buy a synthetic option at the same time (buy low and sell high). The initial cash flow is

$$
\begin{equation*}
\$ 8.00-\$ 7.839=\$ 0.161 \tag{4}
\end{equation*}
$$

and there is no risk at expiration:

|  | Stock Price in 1 Year |  |
| :--- | ---: | ---: |
|  | $\$ 32.903$ | $\$ 59.954$ |
| Written call | $\$ 0$ | $-\$ 19.954$ |
| 0.738 purchased shares | $\$ 24.271$ | $\$ 44.225$ |
| Repay loan of $\$ 22.405$ | $-\$ 24.271$ | $-\$ 24.271$ |
| Total payoff | $\$ 0$ | $\$ 0$ |

## Arbitraging a Mispriced Option (cont'd)

- Suppose that the market price for the option is $\$ 7.5$ instead of $\$ 7.839$ (the option is underpriced).
- We can buy the option and sell a synthetic option at the same time (buy low and sell high). The initial cash flow is

$$
\begin{equation*}
\$ 7.839-\$ 7.5=\$ 0.339 \tag{5}
\end{equation*}
$$

and there is no risk at expiration:

|  | Stock Price in 1 Year |  |
| :--- | ---: | ---: |
|  | $\$ 32.903$ | $\$ 59.954$ |
| Purchased call | $\$ 0$ | $\$ 19.954$ |
| 0.738 short-sold shares | $-\$ 24.271$ | $-\$ 44.225$ |
| Sell T-bill | $\$ 24.271$ | $\$ 24.271$ |
| Total payoff | $\$ 0$ | $\$ 0$ |

## A Remarkable Result

- So far we have not specified the probabilities of the stock going up and down.
- In fact, probabilities were not used anywhere in the option price calculations.
- Since the strategy of holding $\Delta$ shares and $B$ bonds replicates the option whichever way the stock moves, the probability of an up or down movement in the stock is irrelevant for pricing the option.


## The Binomial Solution

- Suppose that the stock has a continuous dividend yield of $\delta$, which is reinvested in the stock. Thus, if you buy one share at time 0 and the length of a period is $h$, at time $h$ you will have $e^{\delta h}$ shares.
- The up and down movements of the stock price reflect the ex-dividend price.
- We can write the stock price as $u S_{0}$ when the stock goes up and $d S_{0}$ when the stock goes down. We can represent the tree for the stock and the option as follows:



## The Binomial Solution (cont'd)

- If the length of a period is $h$, the interest factor per period is $e^{r h}$.
- A successful replicating portfolio will satisfy

$$
\begin{cases}\Delta \times S_{0} \times u \times e^{\delta h}+B \times e^{r h} & =C_{1}^{u}  \tag{6}\\ \Delta \times S_{0} \times d \times e^{\delta h}+B \times e^{r h} & =C_{1}^{d}\end{cases}
$$

- This is a system of two equations in two unknowns $\Delta$ and $B$. Solving for $\Delta$ and $B$ gives

$$
\left\{\begin{array}{ll}
\Delta & =e^{-\delta h \frac{C_{1}^{u}-C_{1}^{d}}{S_{0}(u-d)}}  \tag{7}\\
B & =e^{-r h} \frac{C_{1}^{d} u-C_{1}^{u} d}{u-d}
\end{array}=e^{-r h}\left(C_{1}^{u}-\Delta S_{0} u e^{\delta h}\right)\right.
$$

## The Binomial Solution (cont'd)

- Given the expressions (7) for $\Delta$ and $B$, we can derive a simple formula for the value of the option. The cost of creating the option is the net cash required to buy the shares and bonds. Thus, the cost of the option is

$$
\begin{align*}
C_{0} & =\Delta S_{0}+B \\
& =e^{-r h}\left(C_{1}^{u} \frac{e^{(r-\delta) h}-d}{u-d}+C_{1}^{d} \frac{u-e^{(r-\delta) h}}{u-d}\right) \tag{8}
\end{align*}
$$

- Note that if we are interested only in the option price, it is not necessary to solve for $\Delta$ and $B$; that is just an intermediate step. If we want to know only the option price, we can use equation (8) directly.


## The Binomial Solution (cont'd)

- The assumed stock price movements, $u$ and $d$, should not give rise to arbitrage opportunities. In particular, we require that

$$
\begin{equation*}
d<e^{(r-\delta) h}<u \tag{9}
\end{equation*}
$$

- Note that because $\Delta$ is the number of shares in the replicating portfolio, it can also be interpreted as the sensitivity of the option to a change in the stock price. If the stock prices changes by $\$ 1$, then the option price, $\Delta S+B$, changes by $\Delta$.


## Delta of a Call Option on S\&P 500



Australia 61297778600 Brazil 551130484500 Europe 442073307500 Germany 496992041210 Hong Kong 85229776000 Japan 81332018900 Singapore 6562121000 U.S. 1212318 2000 Copuright 2013 Bloomberg Finance L.P. SN 207036 PDT EMT-7:00 H190-908-1 01-Now-2013 09:25:44

## The Binomial Solution, Special Case: $\delta=0$ and $h=1$

- The solution for $\Delta$ and $B$ reduces to

$$
\left\{\begin{array}{l}
\Delta=\frac{C_{1}^{u}-C_{1}^{d}}{S_{0}(u-d)}  \tag{10}\\
B=e^{-r} \frac{C_{1}^{d} u-C_{1}^{u} d}{u-d}=e^{-r}\left(C_{1}^{u}-\Delta S_{0} u\right)
\end{array}\right.
$$

- The option price further simplifies to

$$
\begin{equation*}
C_{0}=\Delta S_{0}+B=e^{-r}\left(C_{1}^{u} \frac{e^{r}-d}{u-d}+C_{1}^{d} \frac{u-e^{r}}{u-d}\right) \tag{11}
\end{equation*}
$$

Problem 1 Let $S=\$ 100, K=\$ 105, r=8 \%$ (continuously compounded), $T=0.5$, and $\delta=0$. Let $u=1.3, d=0.8$, and the number of binomial periods $n=1$. What are the premium, $\Delta$, and $B$ for a European call?

## Problem 1 Let $S=\$ 100, K=\$ 105, r=8 \%$ (continuously

 compounded), $T=0.5$, and $\delta=0$. Let $u=1.3, d=0.8$, and the number of binomial periods $n=1$. What are the premium, $\Delta$, and $B$ for a European call?Using the formulas given in (7), we calculate the following values:

$$
\begin{aligned}
\Delta & =0.5 \\
B & =-38.4316
\end{aligned}
$$

Call price $=11.5684$

## A Two-Period Binomial Tree

- We can extend the previous example to price a 2 -year option, assuming all inputs are the same as before.

- Note that an up move followed by a down move $\left(S_{2}^{u d}\right)$ generates the same stock price as a down move followed by an up move $\left(S_{2}^{d u}\right)$. This is called a recombining tree.


## A Two-Period Binomial Tree (cont'd)

- To price the option when we have two binomial periods, we need to work backward through the tree.
- Suppose that in period 1 the stock price is $S_{1}^{u}=\$ 59.954$. We can use equation (11) to derive the option price:

$$
\begin{equation*}
C_{1}^{u}=e^{-r}\left(C_{2}^{u u} \frac{e^{r}-d}{u-d}+C_{2}^{u d} \frac{u-e^{r}}{u-d}\right)=\$ 23.029 \tag{12}
\end{equation*}
$$

- Using equations (10), we can also solve for the composition of the replicating portfolio:

$$
\begin{equation*}
\Delta=1, B=-36.925 \tag{13}
\end{equation*}
$$

i.e., buy 1 share of XYZ and borrow $\$ 36.925$ at the risk-free rate, which costs $1 \times \$ 59.954-\$ 36.925=\$ 23.029$.

## A Two-Period Binomial Tree (cont'd)

- Suppose that in period 1 the stock price is $S_{1}^{d}=\$ 32.903$. We can use equation (11) to derive the option price:

$$
\begin{equation*}
C_{1}^{d}=e^{-r}\left(C_{2}^{u d} \frac{e^{r}-d}{u-d}+C_{2}^{d d} \frac{u-e^{r}}{u-d}\right)=\$ 3.187 \tag{14}
\end{equation*}
$$

- Using equations (10), we can also solve for the composition of the replicating portfolio:

$$
\begin{equation*}
\Delta=0.374, B=-9.111 \tag{15}
\end{equation*}
$$

i.e., buy 0.374 shares of XYZ and borrow $\$ 9.111$ at the risk-free rate, which costs $0.374 \times \$ 32.903-\$ 9.111=\$ 3.187$.

## A Two-Period Binomial Tree (cont'd)

- Move backward now at period 0 . The stock price is $S_{0}=41$. We can use equation (11) to derive the option price:

$$
\begin{equation*}
C_{0}=e^{-r}\left(C_{1}^{u} \frac{e^{r}-d}{u-d}+C_{1}^{d} \frac{u-e^{r}}{u-d}\right)=\$ 10.737 \tag{16}
\end{equation*}
$$

- Using equations (10), we can also solve for the composition of the replicating portfolio:

$$
\begin{equation*}
\Delta=0.734, \quad B=-19.337 \tag{17}
\end{equation*}
$$

i.e., buy 0.734 shares of XYZ and borrow $\$ 19.337$ at the risk-free rate, which costs $0.734 \times \$ 41-\$ 19.337=\$ 10.737$.

## A Two-Period Binomial Tree (cont'd)

- The two-period binomial tree with the option price at each node as well as the details of the replicating portfolio is:



## Many Binomial Periods: Three-Period Example

- Once we understand the two-period option it is straightforward to value an option using more than two binomial periods.
- The important principle is to work backward through the tree:



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## Put Options

- We compute put option prices using the same stock price tree and in the same way as call option prices.
- The only difference with an European put option occurs at expiration: Instead of computing the price as $\max [0, S-K]$, we use $\max [0, K-S]$.
- Here is a two-period binomial tree for an European put option with a \$40 strike:


## Put Options (cont'd)



- The proof that the replicating portfolio is self-financing is left as an exercise.


## Put Options (Using the Parity Relationship)

- For non-dividend paying stocks, the basic parity relationship for

European options with the same strike price and time to expiration is

$$
\begin{equation*}
C_{t}-P_{t}=S_{t}-P V(\text { strike price }) \tag{18}
\end{equation*}
$$

- We can use this relationship to find the put price at all nodes:


Problem 2 Let $S=\$ 100, K=\$ 105, r=8 \%$ (continuously compounded), $T=0.5$, and $\delta=0$. Let $u=1.3, d=0.8$, and the number of binomial periods $n=1$. What are the premium, $\Delta$, and $B$ for a European put?

## Problem 2 Let $S=\$ 100, K=\$ 105, r=8 \%$ (continuously

 compounded), $T=0.5$, and $\delta=0$. Let $u=1.3, d=0.8$, and the number of binomial periods $n=1$. What are the premium, $\Delta$, and $B$ for a European put?Using the formulas given in (7), we calculate the following values:

$$
\begin{aligned}
\Delta & =-0.5 \\
B & =62.4513 \\
\text { Put price } & =12.4513
\end{aligned}
$$

## Interview questions

- How many nodes are there in a recombining binomial tree with $N$ time steps?


## Interview questions

- How many nodes are there in a recombining binomial tree with $N$ time steps?
- How many nodes are there in a non-recombining binomial tree with $N$ time steps?


## Interview questions

- How many nodes are there in a recombining binomial tree with $N$ time steps?
- How many nodes are there in a non-recombining binomial tree with $N$ time steps?
- Assume no dividends. The stock price is $\$ 100$. The riskless interest rate is $5 \%$ per annum. Consider a one-year European call option struck at-the-money. If the volatility is zero, what is the call worth?


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## Non-Recombining Binomial Trees

- We show how to modify our binomial tree to deal with path dependent options
- We illustrate the ideas with Asian options (also called Average Rate Options)
- Using a non-recombining binomial tree will mean that at time $t=n$ there will be $2^{n}$ states of the world as opposed to $n+1$ states with recombining trees
- This causes problems as the number of states is growing exponentially


## Asian Option Example

- Let's go back to the previous example with two periods: $S_{0}=\$ 41$, $u=1.4623, d=0.8025, K=40, T=2, h=1$, and $r=0.08$.
- Consider an Asian call option which pays-off the following non-negative amount at maturity:

$$
\begin{equation*}
G_{T}=\max \left[\frac{1}{3} \sum_{k=0}^{2} S_{k}-40,0\right] \tag{19}
\end{equation*}
$$

- The two-period binomial tree with the option price at each node as well as the replicating portfolio is (details will be provided in class):


## Asian Option Example (cont'd)

| 4 | A | B | C | D | E | F | G |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 |  |  |  |  |  |  |  |
| 2 |  | S | 41 |  |  |  |  |
| 3 |  | K | 40 |  |  |  |  |
| 4 |  | u | 1.4623 |  |  |  | 87.6712 |
| 5 |  | d | 0.8025 |  |  |  | 62.8752 |
| 6 |  | r | 0.08 |  |  |  | 22.8752 |
| 7 |  | T | 2 |  | 59.9543 |  |  |
| 8 |  | delta | 0 |  | 50.4772 |  |  |
| 9 |  |  |  |  | 14.1243 |  |  |
| 10 |  | periods | 2 |  | 0.3333 |  | 48.1133 |
| 11 |  | h | 1 |  | -5.8605 |  | 49.6892 |
| 12 |  |  |  |  |  |  | 9.6892 |
| 13 |  | Stock | 41 |  |  |  |  |
| 14 |  | Mean | 41 |  |  |  |  |
| 15 |  | Call | 5.6886 |  |  |  |  |
| 16 |  | Shares | 0.5124 |  |  |  | 48.1133 |
| 17 |  | Cash | -15.3182 |  |  |  | 40.6719 |
| 18 |  |  |  |  |  |  | 0.6719 |
| 19 |  |  |  |  | 32.9025 |  |  |
| 20 |  |  |  |  | 36.9513 |  |  |
| 21 |  |  |  |  | 0.2640 |  |  |
| 22 |  |  |  |  | 0.0310 |  | 26.4043 |
| 23 |  |  |  |  | -0.7544 |  | 33.4356 |
| 24 |  |  |  |  |  |  | 0 |
| 25 |  |  |  |  |  |  |  |
| 26 |  |  |  |  |  |  |  |

## Asian Option Example (cont'd)

- It is left as an exercise to price the option whose payoff depends on the geometric average:

$$
\begin{equation*}
G_{T}=\max \left[\left(\prod_{k=0}^{2} S_{k}\right)^{1 / 3}-40,0\right] \tag{20}
\end{equation*}
$$

- See Chapter 14.2 in McDonald.


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## Uncertainty in the Binomial Model

- A natural measure of uncertainty about the stock return is the annualized standard deviation of the continuously compounded stock return, which we will denote by $\sigma$.
- If we split the year into $n$ periods of length $h$ (so that $h=1 / n$ ), the standard deviation over the period of length $h, \sigma_{h}$, is (assuming returns are uncorrelated over time)

$$
\begin{equation*}
\sigma_{h}=\sigma \sqrt{h} \tag{21}
\end{equation*}
$$

- In other words, the standard deviation of the stock return is proportional to the square root of time.


## Uncertainty in the Binomial Model (cont'd)

- We incorporate uncertainty into the binomial tree by modeling the up and down moves of the stock price. Without uncertainty, the stock price next period must equal:

$$
\begin{equation*}
S_{t+h}=S_{t} e^{(r-\delta) h} \tag{22}
\end{equation*}
$$

- To interpret this, without uncertainty, the rate of return on the stock must be the risk-free rate. Thus, the stock price must rise at the risk-free rate less the dividend yield, $r-\delta$.
- We now model the stock price evolution as

$$
\begin{align*}
& u S_{t}=S_{t} e^{(r-\delta) h} e^{+\sigma \sqrt{h}} \\
& d S_{t}=S_{t} e^{(r-\delta) h} e^{-\sigma \sqrt{h}} \tag{23}
\end{align*}
$$

## Uncertainty in the Binomial Model (cont'd)

- We can rewrite this as

$$
\begin{align*}
u & =e^{(r-\delta) h+\sigma \sqrt{h}} \\
d & =e^{(r-\delta) h-\sigma \sqrt{h}} \tag{24}
\end{align*}
$$

- Return has two parts, one of which is certain $[(r-\delta) h]$, and the other of which is uncertain and generates the up and down stock price moves $(\sigma \sqrt{h})$.
- Note that if we set volatility equal to zero, we are back to (22) and we have $S_{t+h}=u S_{t}=d S_{t}=S_{t} e^{(r-\delta) h}$. Zero volatility does not mean that prices are fixed; it means that prices are known in advance.


## Uncertainty in the Binomial Model (cont'd)

- In our example we assumed that $u=1.4623$ and $d=0.8025$. These correspond to an annual stock price volatility of $30 \%$ :

$$
\begin{align*}
& u=e^{(0.08-0) \times 1+0.3 \times \sqrt{1}}=1.4623 \\
& d=e^{(0.08-0) \times 1-0.3 \times \sqrt{1}}=0.8025 \tag{25}
\end{align*}
$$

- We will use equations (24) to construct binomial trees. This approach (called the forward tree approach) is very convenient because it never violates the no arbitrage restriction

$$
d<e^{(r-\delta) h}<u
$$

## Problem 3 For a stock index, $S=\$ 100, \sigma=30 \%$,

 $r=5 \%, \delta=3 \%$, and $T=3$. Let $n=3$.- What is the price of a European call option with a strike of \$95?
- What is the price of a European put option with a strike of $\$ 95$ ?


## Problem 3 For a stock index, $S=\$ 100, \sigma=30 \%$,

 $r=5 \%, \delta=3 \%$, and $T=3$. Let $n=3$.- What is the price of a European call option with a strike of $\$ 95$ ?


The price of a European call option with a strike of 95 is \$24.0058

- What is the price of a European put option with a strike of $\$ 95$ ?


The price of a European put option with a strike of 95 is \$14.3799

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## Risk-Neutral Pricing

- There is a probabilistic interpretation of the binomial solution for the price of an option (equation 8, restated below):

$$
\begin{equation*}
C_{0}=e^{-r h}\left(C_{1}^{u} \frac{e^{(r-\delta) h}-d}{u-d}+C_{1}^{d} \frac{u-e^{(r-\delta) h}}{u-d}\right) \tag{26}
\end{equation*}
$$

- The terms $\frac{e^{(r-\delta) h}-d}{u-d}$ and $\frac{u-e^{(r-\delta) h}}{u-d}$ sum to 1 and are both positive (this follows from inequality 9).
- Thus, we can interpret these terms as probabilities. Equation (8) can be written as

$$
\begin{equation*}
C_{0}=e^{-r h}\left[p^{*} C_{1}^{u}+\left(1-p^{*}\right) C_{1}^{d}\right] \tag{27}
\end{equation*}
$$

- This expression has the appearance of an expected value discounted at the risk-free rate. Thus, we will call $p^{*} \equiv \frac{e^{(r-\delta) h}-d}{u-d}$ the risk-neutral probability of an increase in the stock price.


## Risk-Neutral Pricing (cont'd)

- Pricing options using risk-neutral probabilities can be done in one step, no matter how large the number of periods. In our example, $p^{*}=0.4256$. For the one-period European call option we have:

| Call Price in 1 Year | Probability |
| :---: | :---: |
| $\$ 19.954$ | 0.4256 |
| $\$ 0$ | 0.5744 |

- The call price is

$$
\begin{equation*}
C_{0}=e^{-0.08}(0.4256 \times \$ 19.954+0.5744 \times \$ 0)=\$ 7.839 \tag{28}
\end{equation*}
$$

## Risk-Neutral Pricing (cont'd)

- For the two-period European call option we have:

| Call Price in 2 Years | Probability |
| :---: | :---: |
| $\$ 47.669$ | $p^{* 2}=0.1811$ |
| $\$ 8.114$ | $2 p^{*}\left(1-p^{*}\right)=0.4889$ |
| $\$ 0$ | $\left(1-p^{*}\right)^{2}=0.3300$ |

- The call price is

$$
\begin{align*}
C_{0} & =e^{-0.08 \times 2}(0.1811 \times \$ 47.669+0.4889 \times \$ 8.114+0.3300 \times \$ 0) \\
& =\$ 10.737 \tag{29}
\end{align*}
$$

- The probability of reaching any given node is the probability of one path reaching that node times the number of paths reaching that node. For example, the probability of reaching the node $S_{2}^{u d}=\$ 48.114$ is $2 p^{*}\left(1-p^{*}\right)$.
- It can be easily verified that the sum of probabilities in the table above is 1 .


## Risk-Neutral Pricing (cont'd)

- For the three-period European call option we have:

| Call Price in 3 Years | Probability |
| :---: | :---: |
| $\$ 88.198$ | $p^{* 3}=0.0771$ |
| $\$ 30.356$ | $3 p^{* 2}\left(1-p^{*}\right)=0.3121$ |
| $\$ 0$ | $3 p^{*}\left(1-p^{*}\right)^{2}=0.4213$ |
| $\$ 0$ | $\left(1-p^{*}\right)^{3}=0.1896$ |

- The call price is

$$
\begin{align*}
C_{0} & =e^{-0.08 \times 3} \sum_{k=0}^{3} \frac{3!}{k!(3-k)!} p^{* k}\left(1-p^{*}\right)^{3-k} \max \left[S_{0} u^{k} d^{3-k}-K, 0\right]  \tag{30}\\
& =\$ 12.799
\end{align*}
$$

- It can be easily verified that the sum of probabilities in the table above is 1 .
- It is left as an exercise to find the price of the two-period European put using risk-neutral probabilities.


## Risk-Neutral Pricing (cont'd)

- For an arbitrary number of periods $n$, the price of an European call option is given by

$$
\begin{equation*}
C_{0}=e^{-r T} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^{* k}\left(1-p^{*}\right)^{n-k} \max \left[S_{0} u^{k} d^{n-k}-K, 0\right] \tag{31}
\end{equation*}
$$

- We will use this formula later on when talking about Black-Scholes: when the number of steps becomes great enough the price of the option appear to approach a limiting value. This value is given by the Black-Scholes formula.

Problem 4 Let $S=\$ 100, \sigma=0.3, r=0.08, T=1$, and $\delta=0$. Use equation (31) to compute the risk-neutral probability of reaching a terminal node and the price at that node for $n=3$. Plot the risk-neutral distribution of year-1 stock prices.

Problem 4 Let $S=\$ 100, \sigma=0.3, r=0.08, T=1$, and $\delta=0$. Use equation (31) to compute the risk-neutral probability of reaching a terminal node and the price at that node for $n=3$. Plot the risk-neutral distribution of year-1 stock prices.

For $n=3, u$ and $d$ are calculated as follows:

$$
\begin{aligned}
& u=e^{(0.08-0) \times 1 / 3+0.3 \times \sqrt{1 / 3}}=1.2212 \\
& d=e^{(0.08-0) \times 1 / 3-0.3 \times \sqrt{1 / 3}}=0.8637 \\
& \text { It follows that } p^{*}=0.4568 \text {, and }
\end{aligned}
$$



## Understanding Risk-Neutral Pricing

- A risk-neutral investor is indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing.
- A risk-averse investor prefers a sure thing to a risky bet with an expected payoff equal to the value of the sure thing.
- Formula (27) suggests that we are discounting at the risk-free rate, even though the risk of the option is at least as great as the risk of the stock.
- Thus, the option pricing formula, equation (27), can be said to price options as if investors are risk-neutral.


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## American Options

- An European option can only be exercised at the expiration date, whereas an American option can be exercised at any time.
- Because of this added flexibility, an American option must always be at least as valuable as an otherwise identical European option:

$$
\begin{aligned}
& C_{\text {Amer }}(S, K, T) \geq C_{E u r}(S, K, T) \\
& P_{\text {Amer }}(S, K, T) \geq P_{\text {Eur }}(S, K, T)
\end{aligned}
$$

- Combining these statements, together with the maximum and minimum option prices (from the first Section), gives us

$$
\begin{aligned}
& S \geq C_{\text {Amer }}(S, K, T) \geq C_{\text {Eur }}(S, K, T) \\
& K \geq P_{\text {Amer }}(S, K, T)\left.\geq P_{\text {Eur }}(S, K, T), S-P V(\text { Div })-P V(K)\right] \\
& \max [0, P V(K)-S+P V(\text { Div })]
\end{aligned}
$$

## American Call

- An American-style call option on a nondividend-paying stock should never be exercised prior to expiration (proof in class).
- For an American call on a dividend-paying stock it might be beneficial to exercise the option prior to expiration (by exercising the call, the owner will be entitled to dividend payments that she would not have otherwise received).
- Consider the previous example, with one exception: $\delta=0.065$, i.e., XYZ stock has a continuous dividend yield of $6.5 \%$ per year.


## American Call (cont'd)

- Because of dividends, early exercise is optimal at the node where the stock price is $\$ 76.982$.



## American Put

- When the underlying stock pays no dividend, a call will not be early-exercised, but a put might be.
- Suppose a company is bankrupt and the stock price falls to zero. Then a put that would not be exercised until expiration will be worth $P V(K)$, which is smaller than $K$ for a positive interest rate.
- Therefore, early exercise would be optimal in order to receive the strike price earlier.
- This can also be shown by using a parity argument.
- Consider the previous example, with $\delta=0.065$.


## American Put (cont'd)

- Early exercise is optimal at the node where the stock price is $\$ 23.187$.
$S_{0}=\$ 41$
$P_{0}=\$ 6.546$
$P_{0, N O}=\$ 6.546$

$P_{0, E X}=\$ 0$$\quad$| $P_{1, N O}^{u}=\$ 2.314$ |
| :--- |
| $P_{1, E X}^{u}=\$ 0$ |$\quad$| $S_{1}^{d}=\$ 30.833$ |
| ---: |
| $P_{1}^{d}=\$ 10.630$ |
| $P_{1, N O}^{d}=\$ 10.630$ |
| $P_{1, E X}^{d}=\$ 9.167$ |

$P_{\cdot, N O}=$ value of put if not exercised
$P C_{\cdot, E X}=$ value of put if exercised

## Problem 5 For a stock index, $S=\$ 100, \sigma=30 \%$,

 $r=5 \%, \delta=3 \%$, and $T=3$. Let $n=3$.- What is the price of an American call option with a strike of $\$ 95$ ?
- What is the price of an American put option with a strike of $\$ 95$ ?


## Problem 5 For a stock index, $S=\$ 100, \sigma=30 \%$,

 $r=5 \%, \delta=3 \%$, and $T=3$. Let $n=3$.- What is the price of an American call option with a strike of $\$ 95$ ?


The price of an American call option with a strike of 95 is \$24.1650

- What is the price of an American put option with a strike of $\$ 95$ ?


The price of an American put option with a strike of 95 is \$15.2593

## Understanding Early Exercise

- In deciding whether to early-exercise an option, the option holder compares the value of exercising immediately with the value of continuing to hold the option.
- Consider the cost and benefits of early exercise for a call option and a put option. By exercising, the option holder


## Call Option

- Receives the stock and therefore receives future dividends
- Bears the interest cost of paying the strike price prior to expiration
- Loses the insurance implicit in the call (the option holder is protected against the possibility that the stock price will be less than the strike price at expiration).


## Put Option

- Dividends are lost by giving up the stock
- Receives the strike price sooner rather than later
- Loses the insurance implicit in the put (the option holder is protected against the possibility that the stock price will be more than the strike price at expiration)


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Real Option $=$ the right, but not the obligation, to make a particular business decision

- The application of derivatives theory to the operation and valuation of real investment projects
- A call option is the right to pay a strike price to receive the present value of a stream of future cash flows
- An investment project is the right to pay an investment cost to receive the present value of a future cash flow stream:

| Investment Project | Call Option |
| ---: | :--- |
| Investment Cost | $=$ Strike Price |
| Present Value of Project | $=$ Price of Underlying Asset |

## Investment Under Uncertainty

- A project requires an initial investment of $\$ 100$.

$$
\begin{equation*}
\Rightarrow K=100 \tag{32}
\end{equation*}
$$

- The project is expected to generate a perpetual cash flow stream, with a first cash flow $\$ 18$ in one year, expected to grow at $3 \%$ annually. Assume a discount rate of $15 \%$.

$$
\Rightarrow\left\{\begin{array}{l}
\text { Perpetual growing annuity } \Rightarrow P V=\frac{\$ 18}{0.15-0.03}=\$ 150  \tag{33}\\
\text { Static NPV }=\$ 150-\$ 100=\$ 50 \\
\text { Cont. compounded dividend yield } \delta=\ln \left(1+\frac{\$ 18}{\$ 150}\right)=0.1133
\end{array}\right.
$$

- The cont. compounded risk-free rate is $r=6.766 \%$. The cash flows of the project are normally distributed with a volatility of $\sigma=50 \%$.

$$
\Rightarrow\left\{\begin{array}{l}
u=e^{(r-\delta)+\sigma}=1.5751  \tag{34}\\
d=e^{(r-\delta)-\sigma}=0.5795
\end{array}\right.
$$

## Investment Under Uncertainty (cont.)

- Suppose the project can be delayed: we can decide whether to accept the project at time 0,1 , or 2 .
- Should the project be accepted? If yes, when?
- Trade-off:
- Invest today and get an NPV of $\$ 50$ ?
or
- Delay project and avoid possible future loss?


## Binomial tree for project cash flows \& project value



- The risk-neutral probability of the project value increasing in any period, $p^{*}$, is given by

$$
\begin{equation*}
p^{*}=\frac{e^{r-\delta}-d}{u-d}=0.3775 \tag{35}
\end{equation*}
$$

## Value of the investment option



- Notice that the initial value of the project option is $\$ 55.66$, which is greater than the static NPV of $\$ 50$.
- If we invest immediately, the project is worth $\$ 50$. The ability to wait increases that value by $\$ 5.66$.
- Exercise: try a volatility of $\sigma=30 \%$. What do you observe?


## Valuing the option to expand capacity

- Desautels Industries is considering building a plant. The plant will generate cash flows two years from now. The cash flows will be:
- $\$ 200$ million following two good years
- $\$ 150$ million following one good year and one bad year
- \$100 million following two bad years
- The initial cost of the plant is $\$ 140$ million
- After one year, Desautels Industries has the option to double the plant's capacity by investing another $\mathbf{\$ 1 4 0}$ million
- Assume a risk-free rate is $6 \%$ per year. The risk-neutral probability of a good year is $p^{*}=0.53$.
- Compute the value of building the plant under two scenarios:

1. The option to double the plant's capacity is ignored
2. The option to double the plant's capacity is not ignored
(solution in class)

## Real Options in Practice

- The decision about whether and when to invest in a project $\sim$ call option
- The ability to shut down, restart, and permanently abandon a project $\sim$ project + put option
- Strategic options: the ability to invest in projects that may give rise to future options $\sim$ compound option
- Flexibility options: the ability to switch between inputs, outputs, or technologies $\sim$ rainbow option


## Final Thoughts: Is the Binomial Model Realistic?

- Volatility is constant
- There is ample evidence that volatility changes over time
- Large stock price movements do not occur
- It appears that on occasion stocks move by a large amount (jumps)
- Returns are independent over time
- There is strong evidence that stock returns are correlated across time, with positive correlations at the short to medium term (momentum) and negative correlation at long horizons (reversal).
- Continuous dividend yield $\delta$
- Stocks pay dividends in discrete lumps, quarterly or annually
- In addition, over short horizons it is frequently possible to predict the amount of the dividend
- The binomial tree can be adjusted to accommodate this case (see Chapter 11.4 in McDonald)
- Problem Set 2 is available


# Financial Derivatives FINE 448 <br> 4. Black-Scholes: Formula and the Greeks 

Daniel Andrei

## (4n McGill

Winter 2021
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## Schedule: Mon \& Wed, 8:35am-9:55am (Section 1) and 4:05pm-5:25pm (Section 2), online (recorded)

| Class \# | Day | Class / Other activities |
| :---: | :--- | :--- |
| 1 | Jan. 11, Mon | Class |
| 2 | Jan. 13, Wed | Class |
| 3 | Jan. 18, Mon | Class |
| 4 | Jan. 20, Wed | Class |
| 5 | Jan. 25, Mon | Class |
| 6 | Jan. 27, Wed | Class |
| 7 | Feb. 1, Mon | Class |
| 8 | Feb. 3, Wed | Class |
| 9 | Feb. 8, Mon | Class |
| 10 | Feb. 10, Wed | Class |
| 11 | Feb. 15, Mon | Class |
| 12 | Feb. 17, Wed | Class |
| 13 | Feb. 22, Mon | Class |
| $14^{* * *}$ | Feb. 24, Wed | MIDTERM (details TBD) |
| $* * \pi=$ important dates |  |  |


| Class \# | Day | Class / Other activities |
| :---: | :--- | :--- |
| - | Mar. 1, Mon | No class (Reading week) |
| - | Mar. 3, Wed | No class (Reading week) |
| 15 | Mar. 8, Mon | Class |
| 16 | Mar. 10, Wed | Class |
| 17 | Mar. 15, Mon | Class |
| 18 | Mar. 17, Wed | Class |
| 19 | Mar. 22, Mon | Class |
| 20 | Mar. 24, Wed | Class |
| 21 | Mar. 29, Mon | Class |
| 22 | Mar. 31, Wed | Class |
| - | Apr. 5, Mon | No class (Easter) |
| 23 | Apr. 7, Wed | Class |
| 24 | Apr. 12, Mon | Class |
| 25 | Apr. 14, Wed | Class |
| 26 | Apr. 15, Thu | Class (makeup) |

- Makeup class: Apr. 15, Thu.
- Midterm: Feb. 24, Wed.
- Final Exam: Apr. 26, Mon., 2:00 PM - Apr. 29, Thu., 2:00 PM 3 hours, details TBD


## Discrete Time vs Continuous Time

- We refer to the structure of the binomial model as discrete time, which means that time moves in distinct increments
- This is much like looking at a calendar and observing only months, weeks, or days
- We know that time moves forward at a rate faster than one day at a time: hours, minutes, seconds, fractions of seconds, and fractions of fractions of seconds
- When we talk about time moving in the tiniest increments, we are talking about continuous time.

| Discrete time world | Continuous time world |
| :---: | :---: |

Binomial model $\quad$ Black-Scholes model

## Discrete Time vs Continuous Time



## Discrete Time vs Continuous Time



## Discrete Time vs Continuous Time



## Discrete Time vs Continuous Time



## Discrete Time vs Continuous Time



## Discrete Time vs Continuous Time



## The Limiting Case of the Binomial Formula

- An obvious objection to the binomial calculations thus far is that the stock can only have a few different values at expiration. It seems unlikely that the option price calculation will be accurate.
- The solution to this problem is to divide the time to expiration into more periods, generating a more realistic tree.
- To illustrate how to do this, we will re-examine the 1-year European call option, which has a $\$ 40$ strike and initial stock price $\$ 41$.
- Let there be 3 binomial periods. Since it is a 1 -year call, this means that the length of a period is $h=1 / 3$. We will assume that other inputs stay the same, so $r=0.08$ and $\sigma=0.3$.


## The Limiting Case of the Binomial Formula (cont'd)

- Since the length of the binomial period is shorter, $u$ and $d$ are closer to 1 than before ( 1.2212 and 0.8637 as opposed to 1.4623 and 0.8025 with $h=1$ ).
- The risk-neutral probability of the stock price going up in a period is

$$
p^{*}=\frac{e^{(0.08-0) \times 1 / 3}-0.8637}{1.2212-0.8637}=0.4568
$$

- The binomial model implicitly assigns probabilities to the various nodes. The risk-neutral probability for each final period node, together with the call value, is:

| Call Price in 1 Year (3 periods) | Probability |
| :---: | :---: |
| $\$ 34.678$ | $p^{* 3}=0.0953$ |
| $\$ 12.814$ | $3 p^{* 2}\left(1-p^{*}\right)=0.34$ |
| $\$ 0$ | $3 p^{*}\left(1-p^{*}\right)^{2}=0.4044$ |
| $\$ 0$ | $\left(1-p^{*}\right)^{3}=0.1603$ |

## The Limiting Case of the Binomial Formula (cont'd)

- Thus, the price of the European call option is given by

$$
\begin{aligned}
C_{0} & =e^{0.08 \times 1}(0.0953 \times \$ 34.678+0.34 \times \$ 12.814) \\
& =\$ 7.0739
\end{aligned}
$$

- We can vary the number of binomial steps, holding fixed the time to expiration, $T$. The general formula is

$$
\begin{equation*}
C_{0}=e^{-r T} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^{* k}\left(1-p^{*}\right)^{n-k} \max \left[S_{0} u^{k} d^{n-k}-K, 0\right] \tag{1}
\end{equation*}
$$

(we also need to modify $u, d$, and $p^{*}$ at each time).

## The Limiting Case of the Binomial Formula (cont'd)

- The following table computes binomial call option prices, using the same inputs as before.

| Number of steps (n) | Binomial Call Price (\$) |
| :---: | :---: |
| 1 | $\$ 7.839$ |
| 2 | $\$ 7.162$ |
| 3 | $\$ 7.074$ |
| 4 | $\$ 7.160$ |
| 10 | $\$ 7.065$ |
| 50 | $\$ 6.969$ |
| 100 | $\$ 6.966$ |
| 500 | $\$ 6.960$ |
| $\infty$ | $\$ 6.961$ |

- Changing the number of steps changes the option price, but once the number of steps becomes great enough we appear to approach a limiting value for the price, given by the Black-Scholes formula.


## The Limiting Case of the Binomial Formula (cont'd)

- This can be seen graphically below:


Problem 12.2: Using the BinomCall function, compute the binomial approximations for the following call option:
$S_{0}=\$ 41, K=\$ 40, \sigma=0.3, r=0.08, T=0.25$ (3 months), and $\delta=0$. Be sure to compute prices for $n=8,9,10,11$, and 12 . What do you observe about the behavior of the binomial approximation?

Problem 12.2: Using the BinomCall function, compute the binomial approximations for the following call option: $S_{0}=\$ 41, K=\$ 40, \sigma=0.3, r=0.08, T=0.25$ (3 months), and $\delta=0$. Be sure to compute prices for $n=8,9,10,11$, and 12 . What do you observe about the behavior of the binomial approximation?

| N | Call Price |
| :--- | :--- |
| 8 | 3.464 |
| 9 | 3.361 |
| 10 | 3.454 |
| 11 | 3.348 |
| 12 | 3.446 |

The observed values are slowly converging towards the Black-Scholes value (3.399). Please note that the binomial solution oscillates as it approaches the Black-Scholes value.

## Lognormality and the Binomial Model

- The lognormal distribution is the probability distribution that arises from the assumption that continuously compounded returns on the stock are normally distributed.
- In a binomial tree, as we increase the number of periods until expiration, continuously compounded returns approach a normal distribution.
- We can plot probabilities of outcomes (of stock returns) from the binomial tree for different values of $n(2,3,6$, and 10), as shown in the following figure.
- The 10-period binomial tree approaches fairly well the normal distribution.


## Lognormality and the Binomial Model (cont'd)



Problem 11.13: Let $S=\$ 100, \sigma=0.30, r=0.08, t=1$, and $\delta=0$. Use equation (1) to compute the probability of reaching a terminal node and the price at that node and plot the risk-neutral distribution of year-1 stock prices for $n=50$. What is the risk-neutral probability that $S_{1}<\$ 80$ ? $S_{1}>\$ 120$ ?

## Problem 11.13: Let $S=\$ 100, \sigma=0.30, r=0.08, t=1$,

 and $\delta=0$. Use equation (1) to compute the probability of reaching a terminal node and the price at that node and plot the risk-neutral distribution of year-1 stock prices for $n=50$. What is the risk-neutral probability that $S_{1}<\$ 80$ ? $S_{1}>\$ 120$ ?For $n=50$, we have $u=1.0450$, $d=0.9600$, and $p^{*}=0.4894$. We get the following diagram:
To obtain the required probabilities we sum all probabilities for which the final stock price is below 80 or above 120 respectively. We obtain

$$
\begin{aligned}
\operatorname{Pr}(S<80) & =0.2006 \\
\operatorname{Pr}(S>120) & =0.2829
\end{aligned}
$$



## Black-Scholes Assumptions

- Assumptions about stock return distribution
- Continously compounded returns on the stocks are normally distributed and independent over time
- The volatility of continuously compounded returns is known and constant
- Future dividends are known, either as dollar amount or as a fixed dividend yield
- Assumptions about the economic environment
- The risk-free rate is known and constant
- There are no transaction costs or taxes
- It is possible to short-sell costlessly and to borrow at the risk-free rate


## Inputs in the Binomial Model and in Black-Scholes

- There are seven inputs to the binomial model and six inputs to the Black-Scholes model:

| Inputs | Binomial <br> Model | Black- <br> Scholes |
| :--- | :--- | :--- |
| Current price of the stock, $S_{0}$ | $\checkmark$ | $\checkmark$ |
| Strike price of the option, $K$ | $\checkmark$ | $\checkmark$ |
| Volatility of the stock, $\sigma$ | $\checkmark$ | $\checkmark$ |
| Continuously compounded risk-free <br> interest rate, $r$ | $\checkmark$ | $\checkmark$ |
| Time to expiration, $T$ | $\checkmark$ | $\checkmark$ |
| Dividend yield on the stock, $\delta$ | $\checkmark$ | $\checkmark$ |
| Number of binomial periods, $n$ | $\checkmark$ |  |

## Convergence from binomial tree to Black-Scholes

- See Appendix of Chapter 12 in Hull (8 $8^{\text {th }}$ edition)
- The binomial price is

$$
\begin{align*}
C_{0} & =e^{-r T} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} p^{* k}\left(1-p^{*}\right)^{n-k} \max \left[S_{0} u^{k} d^{n-k}-K, 0\right]  \tag{2}\\
& =e^{-r T} \sum_{k>\alpha} \frac{n!}{(n-k)!k!} p^{* k}\left(1-p^{*}\right)^{n-k}\left(S_{0} u^{k} d^{n-k}-K\right)  \tag{3}\\
& =e^{-r T}\left(S_{0} U_{1}-K U_{2}\right) \tag{4}
\end{align*}
$$

where

$$
\begin{equation*}
\alpha=\frac{n}{2}-\frac{\ln \left(S_{0} / K\right)+(r-\delta) T}{2 \sigma \sqrt{T / n}} \tag{5}
\end{equation*}
$$

## Convergence from binomial tree to Black-Scholes (cont'd)

- The term

$$
\begin{equation*}
U_{2}=\sum_{k>\alpha} \frac{n!}{(n-k)!k!} p^{* k}\left(1-p^{*}\right)^{n-k} \tag{6}
\end{equation*}
$$

represents the probability of the option being in the money at maturity. When $n \rightarrow \infty$, this probability tends to

$$
\begin{equation*}
U_{2}=N\left(\frac{n p^{*}-\alpha}{\sqrt{n p^{*}\left(1-p^{*}\right)}}\right) \tag{7}
\end{equation*}
$$

- Replace $\alpha, p^{*}, u$, and $d$ in Equation (7) to obtain:

$$
\begin{equation*}
U_{2}=N(\underbrace{\frac{\ln \left(S_{0} / K\right)+\left(r-\delta-\sigma^{2} / 2\right) T}{\sigma \sqrt{T}}}_{d_{2}}) \tag{8}
\end{equation*}
$$

## Convergence from binomial tree to Black-Scholes (cont'd)

- Take now the term

$$
\begin{equation*}
U_{1}=\sum_{k>\alpha} \frac{n!}{(n-k)!k!}\left(p^{*} u\right)^{k}\left[\left(1-p^{*}\right) d\right]^{n-k} \tag{9}
\end{equation*}
$$

and define

$$
\begin{equation*}
q \equiv \frac{p^{*} u}{p^{*} u+\left(1-p^{*}\right) d}=\frac{p^{*} u}{e^{(r-\delta) T / n}} \tag{10}
\end{equation*}
$$

It then follows that

$$
\begin{equation*}
1-q=\frac{\left(1-p^{*}\right) d}{e^{(r-\delta) T / n}} \tag{11}
\end{equation*}
$$

and thus

$$
\begin{equation*}
U_{1}=e^{(r-\delta) T} \sum_{k>\alpha} \frac{n!}{(n-k)!k!} q^{k}(1-q)^{n-k} \tag{12}
\end{equation*}
$$

## Convergence from binomial tree to Black-Scholes (cont'd)

- This looks like (6), except that we have $q$ instead of $p^{*}$. When $n \rightarrow \infty$, this probability tends to

$$
\begin{equation*}
U_{1}=e^{(r-\delta) T} N\left(\frac{n q-\alpha}{\sqrt{n q(1-q)}}\right) \tag{13}
\end{equation*}
$$

- Replace $q, \alpha, p^{*}, u$, and $d$ in Equation (13) to obtain (proof in class):

$$
\begin{equation*}
U_{1}=e^{(r-\delta) T} N(\underbrace{\frac{\ln \left(S_{0} / K\right)+\left(r-\delta+\sigma^{2} / 2\right) T}{\sigma \sqrt{T}}}_{d_{1}}) \tag{14}
\end{equation*}
$$

- We obtain the Black-Scholes formula:

$$
\begin{align*}
C_{0} & =e^{-r T}\left(S_{0} U_{1}-K U_{2}\right)  \tag{15}\\
& =S_{0} e^{-\delta T} N\left(d_{1}\right)-K e^{-r T} N\left(d_{2}\right) \tag{16}
\end{align*}
$$

## Black-Scholes Formula for a European Call Option

- The Black-Scholes formula for a European call option on a stock that pays dividends at the continuous rate $\delta$ is

$$
\begin{equation*}
C_{0}\left(S_{0}, K, \sigma, r, T, \delta\right)=S_{0} e^{-\delta T} N\left(d_{1}\right)-K e^{-r T} N\left(d_{2}\right) \tag{17}
\end{equation*}
$$

where

$$
\begin{align*}
& d_{1}=\frac{\ln \left(\frac{S_{0}}{K}\right)+\left(r-\delta+\frac{1}{2} \sigma^{2}\right) T}{\sigma \sqrt{T}}  \tag{18}\\
& d_{2}=d_{1}-\sigma \sqrt{T} \tag{19}
\end{align*}
$$

- $N(x)$ is the cumulative normal distribution function, which is the probability that a number randomly drawn from a standard normal distribution will be less than $x$.


## Black-Scholes Formula for a European Call Option (cont'd)

- Let $S_{0}=\$ 41, K=\$ 40, \sigma=0.3, r=0.08, T=1$ year, and $\delta=0$. Computing the Black-Scholes call price, we obtain

$$
\begin{aligned}
C_{0} & =\$ 41 \times e^{-0} \times N(0.49898)-\$ 40 \times e^{-0.08} \times N(0.19898) \\
& =\$ 41 \times e^{-0} \times 0.6911-\$ 40 \times e^{-0.08} \times 0.5789 \\
& =\$ 6.961
\end{aligned}
$$

- Note that this result corresponds to the limit obtained from the binomial model.


## Black-Scholes Formula for a European Call Option (cont'd)

- The following figure plots Black-Scholes call option prices (today and at expiration) for stock prices ranging from $\$ 20$ to $\$ 60$.



## A Remarkable Result

- In the binomial model, we have not specified the probabilities of the stock going up and down (which would give us the expected return of the stock).
- The expected return on the stock does not appear in the Black-Scholes formula either.
- This raises a question: Consider a stock with a higher beta (and hence having a higher expected return). A call option on this stock would have a higher probability of settlement in-the-money, hence a higher option price. Why is this not the case?


## A Remarkable Result (cont'd)

- The Black-Scholes formula (17) provides the answer. This formula shows the composition of the replicating portfolio, which in the binomial case was

$$
C_{0}=\Delta S_{0}+B
$$

- We can easily identify $\Delta$ (the position in the risky asset) and $B$ (the dollar amount of borrowing or lending) in the Black-Scholes formula:

$$
\begin{align*}
\Delta & =e^{-\delta T} N\left(d_{1}\right)  \tag{20}\\
B & =-K e^{-r T} N\left(d_{2}\right) \tag{21}
\end{align*}
$$

## A Remarkable Result (cont'd)

- If $\beta_{S}$ is the stock beta, then the option beta is

$$
\begin{equation*}
\beta_{\text {Option }}=\frac{\Delta S_{0}}{\Delta S_{0}+B} \beta_{S} \tag{22}
\end{equation*}
$$

- A high stock beta implies a high option beta, so the discount rate for the expected payoff of the option is correspondingly greater.
- The net result-one of the key insights from the Black-Scholes analysis-is that beta is irrelevant: The larger average payoff to options on high beta stocks is exactly offset by the larger discount rate.


## Black-Scholes Formula for a European Put Option

- The Black-Scholes formula for a European put option is

$$
\begin{equation*}
P_{0}\left(S_{0}, K, \sigma, r, T, \delta\right)=-S_{0} e^{-\delta T} N\left(-d_{1}\right)+K e^{-r T} N\left(-d_{2}\right) \tag{23}
\end{equation*}
$$

where $d_{1}$ and $d_{2}$ are given by equations (18) and (19).

- Put-call parity must hold:

$$
\begin{equation*}
P_{0}\left(S_{0}, K, \sigma, r, T, \delta\right)=C_{0}\left(S_{0}, K, \sigma, r, T, \delta\right)+K e^{-r T}-S e^{-\delta T} \tag{24}
\end{equation*}
$$

- This follows from the formulas (17) and (23), together with the fact that for any $x, N(-x)=1-N(x)$.


## Black-Scholes Formula for a European Put Option (cont'd)

- Let $S_{0}=\$ 41, K=\$ 40, \sigma=0.3, r=0.08, T=1$ year, and $\delta=0$.

Computing the Black-Scholes put price, we obtain

$$
\begin{aligned}
P_{0} & =-\$ 41 \times e^{-0} \times N(-0.49898)+\$ 40 \times e^{-0.08} \times N(-0.19898) \\
& =-\$ 41 \times e^{-0} \times 0.3089+\$ 40 \times e^{-0.08} \times 0.4211 \\
& =\$ 2.886
\end{aligned}
$$

- In the binomial model, if we fix the number of periods to $n=500$, we obtain a price of $\$ 2.885$.
- Computing the price using put-call parity (equation 24) yields

$$
\begin{aligned}
P_{0} & =\$ 6.961+\$ 40 e^{-0.08}-\$ 41 \\
& =\$ 2.886
\end{aligned}
$$

## Black-Scholes Formula for a European Put Option (cont'd)

- The following figure plots Black-Scholes put option prices (today and at expiration) for stock prices ranging from $\$ 20$ to $\$ 60$.



## Problem 12.20: Let $S=\$ 100, K=\$ 90, \sigma=30 \%$, $r=8 \%, \delta=5 \%$, and $T=1$.

a. What is the Black-Scholes call price?
b. Now price a put where $S=\$ 90, K=\$ 100, \sigma=30 \%, r=5 \%$, $\delta=8 \%$, and $T=1$.

Problem 12.20: Let $S=\$ 100, K=\$ 90, \sigma=30 \%$, $r=8 \%, \delta=5 \%$, and $T=1$.
a. What is the Black-Scholes call price?


The Black-Scholes call price is \$17.70
b. Now price a put where $S=\$ 90, K=\$ 100, \sigma=30 \%, r=5 \%$, $\delta=8 \%$, and $T=1$.


The Black-Scholes put price is \$17.70

## Precursors to Black \& Scholes (1973)

Most previous work on option valuation was expressed in terms of warrants and OTC options because listed options did not trade in modern markets until April 1973. In the Bible, the first book, Genesis, describes the supposed first option. Castelli (1877), Bachlier (1900), Sprenkle (1961), Ayres (1965), Boness (1964), Samuelson (1965), Baumol, Malkiel, and Quandt (1966), and Chen (1970) all produced valuation formulas of the same general form. These formulas were incomplete because they all involved arbitrary parameters and they did not include the idea of a self-financing riskless hedge. A few of these examples are presented below.

## 1. Bachlier (1900)



$$
C(S, T)=S N\left(\frac{S-X}{\sigma \sqrt{T}}\right)-X N\left(\frac{S-X}{\sigma \sqrt{T}}\right)-\sigma \sqrt{T}\left(\frac{X-S}{\sigma \sqrt{T}}\right)
$$

Bachlier's formulation permits both negative underlying security and option prices and does not account for the time value of money. As will be shown this formulation gives very small option premium.

## 2. Sprenkle Formula (1961)

$$
\begin{aligned}
& C(S, T)=e_{\uparrow}^{\rho T} S N\left(d_{1}\right)-(1-A) X N\left(d_{2}\right) \\
& d_{1}=\frac{1}{\sigma \sqrt{T}}\left[\ln \left(\frac{e^{\rho T} S}{X}\right)+\frac{1}{2} \sigma^{2} T\right] \text { and } d_{2}=d_{1}-\sigma \sqrt{T}
\end{aligned}
$$

Sprenkle's equation is similar to Black Scholes but without the riskless hedge or discounting. Instead rho ( $\rho$ ) is the average growth rate of the share price and A corresponds to the degree of risk aversion, and both are very difficult to estimate. This formulation produces a very large option premium.

## 3. Boness Formula (1964)

$$
\begin{aligned}
& \underline{C(S, T)}=S N\left(d_{1}\right)-X e_{\&}^{-\rho T} N\left(d_{2}\right) \\
& d_{1}=\frac{1}{\sigma \sqrt{T}}\left[\ln \left(\frac{S}{X}\right)+\left(\rho+\frac{1}{2} \sigma^{2}\right) T\right] \text { and } d_{2}=d_{1}-\sigma \sqrt{T}
\end{aligned}
$$

Boness accounted for the time value of money by discounting exercise proceeds by the expected rate of return on the stock. This formulation gives much better option values but requires the expected return on the stock.

## 4. Samuelson Formula (1965)

$$
\begin{gathered}
C(S, T)=S e^{-(\rho-\alpha) T} N\left(d_{1}\right)-X e^{-a T} N\left(d_{2}\right) \\
d_{1}=\frac{1}{\sigma \sqrt{T}}\left[\ln \left(\frac{S}{X}\right)+\left(\rho+\frac{1}{2} \sigma^{2}\right) T\right] \text { and } d_{2}=d_{1}-\sigma \sqrt{T}
\end{gathered}
$$

Samuelson allowed for the option and the stock to have different levels of risk and thus different expected growth rates. He defined rho $(\rho)$ and alpha ( $\alpha$ ) as the average rate of growth of the stock price and the option price, respectively. Rho and alpha are very difficult to estimate.
5. Samuelson and Merton (1969)


0

They treated the option price as a function of the stock price, and reasoned that the discount rate must be determined by the requirement that investors hold both the stock and the option. Their formula depended on the utility function assumed for the "representative" investor, and involved the sum of many conditional terms.

## 6. Thorpe and Kassouf (Beat the Market 1967)

They graphically approximated warrant prices using linear regression to determine the exponent of a non-linear approximating function. They did not realize that in a no-arbitrage situation the expected return on the hedged position must be the riskless rate. Furthermore, no direct measure of volatility entered their methodology.

## 7. Black and Scholes (JPE 1973)

$$
\begin{aligned}
& C(S, T)=S N\left(d_{1}\right)-X e^{-\frac{\downarrow}{T}} N\left(d_{2}\right) \\
& d_{1}=\frac{1}{\sigma \sqrt{T}}\left[\ln \left(\frac{S}{X}\right)+\left(r+\frac{1}{2} \sigma^{2}\right) T\right] \text { and } d_{2}=d_{1}-\sigma \sqrt{T}
\end{aligned}
$$

Black \& Scholes formula is similar to Boness (1964). However, their breakthrough (with the help of Merton, BS JPE, footnote 3) was to realize that the return on the hedge position must be equivalent to the return on the riskless asset. They used a two security portfolio which did not include the self-financing bond. Merton (Bell Journal, 1973) proved that a self-financing three security portfolio ( $\mathrm{w}_{\mathrm{C}} \mathrm{C}+\mathrm{w}_{\mathrm{S}} \mathrm{S}+\mathrm{w}_{\mathrm{B}} \mathrm{B}$ ) would replicate the option. This correct formula, its intuition, and other accomplishments allowed Scholes and Merton (Black had died) to win the Nobel Prize in Economics in 1996.
"This was pointed out to us by Robert Merton"

## WSJ, Feb 2014: Big Banks Take Hits On Trusty Oil Hedge

## A Crude Hedge

Each year, Wall Street banks enter potentially lucrative deals to guarantee Mexico's state oil company a minimum price for its crude exports. Here's how it works:
 Energy Information Administration (inventories)

## WSJ, Feb 2014: Big Banks Take Hits On Trusty Oil Hedge



## Reuters, Oct 15, 2014: Banks rush to hedge oil option deltas, accelerating rout

- Wall Street banks have scrambled to neutralize their exposure to big oil options trades.
- Banks have written protection to companies and they sell futures contracts to offset option deals that are unexpectedly in the money.
- Oil producers seek to hedge their production by buying put options with strikes $\$ 75$ to $\$ 85$.
- As futures prices approach the these strike levels, big banks that have sold put options (or similar hedges) to oil producers are forced to protect themselves through delta hedging: they sell futures to remain market neutral.
- The delta hedging selling was cited by several traders as a factor behind Tuesday's rapid swoon in prices.


## Market-Maker Risk

- A market-maker stands ready to sell to buyers and to buy from sellers.
- Without hedging, an active market-maker will have an arbitrary position generated by fulfilling customer orders. This arbitrary portfolio has uncontrolled risk.
- Consequently, market-makers attempt to hedge the risk of their position.
- We will se here how they do so.


## Market-Maker Risk (cont'd)

- Suppose a customer wishes to buy 100 European call options with maturity of 91 days. The market-maker fills this order by selling 100 call options. To be specific, suppose that $S=\$ 41, K=\$ 40, \sigma=0.3$, $r=0.08$ (continuously compounded), and $\delta=0$. We will let $T$ denote the expiration time of the option and $t$ the present, so time to expiration is $T-t$. Thus, $T-t=91 / 365=0.249$.
- Suppose that the market-maker does not hedge the written option (naked position) and the stock has the following evolution over the next 5 days:

| Day | Stock (\$) |
| :---: | :---: |
| 0 | 41 |
| 1 | 42.5 |
| 2 | 39.5 |
| 3 | 37 |
| 4 | 40 |
| 5 | 40 |

## Market-Maker Risk (cont'd)

- We can measure the profit of the market-maker by marking-to-market the position: if we liquidated the position today, what would be the gain or loss?

| Day | Stock (\$) | Call <br> Position (\$) | Daily <br> Profit (\$) |
| :--- | :--- | :--- | :--- |
| 0 | 41 | -339.47 |  |
| 1 | 42.5 | -441.04 | -101.57 |
| 2 | 39.5 | -246.31 | 194.73 |
| 3 | 37 | -129.49 | 116.82 |
| 4 | 40 | -271.04 | -141.55 |
| 5 | 40 | -269.27 | 1.77 |

## Market-Maker Risk (cont'd)



## Delta-Hedging

- Suppose the market-maker hedges the position with shares. At time 0 , the delta of a call at a stock price of $\$ 41$ is 0.645 .
- This suggests that a $\$ 1$ increase in the stock price should increase the value of the option by approximatively $\$ 0.645$.
- The market-maker takes an offsetting position in shares, position that hedges the fluctuations in the option price. We say that such a position is delta-hedged.
- Then, the market-maker rebalances the portfolio each day, by computing the new delta of the call.
- The following table summarizes delta, the number of purchased shares, the net investment, and profit for each day for 5 days (interest expenses are ignored for simplicity).


## Delta-Hedging (cont'd)

| Day | Stock <br> $\mathbf{( \$ )}$ | Option <br> Delta | Stock <br> Position <br> (\# shares) $)$ | Daily <br> Profit <br> (Call) | Daily <br> profit <br> (Shares) | Daily <br> profit <br> (Total) |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 41 | 0.645 | 64.54 |  |  |  |
| 1 | 42.5 | 0.730 | 73.03 | -101.57 | 96.81 | -4.77 |
| 2 | 39.5 | 0.548 | 54.81 | 194.73 | -219.10 | -24.38 |
| 3 | 37 | 0.373 | 37.27 | 116.82 | -137.02 | -20.20 |
| 4 | 40 | 0.581 | 58.06 | -141.55 | 111.82 | -29.73 |
| 5 | 40 | 0.580 | 58.01 | 1.77 | 0.00 | 1.77 |

## Delta-Hedging (cont'd)



-     - Naked Position
$\rightarrow$ Delta Hedged
- Delta hedging prevents the position from reacting to small changes in the underlying stock. For large changes, we need to take into account the fluctuations in delta.


## WFC US <EQUITY> OSA <GO> (not delta-hedged)

## <HELP> for explanation.



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## WFC US <EQUITY> OSA <GO> (delta-hedged)

## <HELP> for explanation.



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## Option Greeks

- Option Greeks are formulas that express the change in the option price when an input to the formula changes, taking as fixed all other inputs.
- They are used to assess risk exposures. For example:
- A market-making bank with a portfolio of options would want to understand its exposure to stock price changes, interest rates, volatility, maturity, etc.
- A portfolio manager wants to know what happens to the value of a portfolio of stock index options if there is a change in the level of the stock index.
- An options investor would like to know how interest rate changes and volatility changes affect profit and loss.


## Option Greeks (cont'd)

- Before providing detailed definition of the Greeks, let's have some intuition on how changes in inputs affect option prices:
Change in input $\quad$ Change in call price $\quad$ Change in put price

| $S_{t} \uparrow$ | $C_{t} \uparrow$ | $P_{t} \downarrow$ |
| :---: | :---: | :---: |
| $\sigma \uparrow$ | $C_{t} \uparrow$ | $P_{t} \uparrow$ |
| $T-t \downarrow(t \uparrow)$ | $C_{t}$ generally $\downarrow$ | $P_{t}$ ambiguous |
| $r \uparrow$ | $C_{t} \uparrow$ | $P_{t} \downarrow$ |
| $\delta \uparrow$ | $C_{t} \downarrow$ | $P_{t} \uparrow$ |

Table 1: Changes in Black-Scholes inputs and their effect on option prices.

- An increase in the stock price $\left(S_{t}\right)$ raises the chance that the call will be exercised, thus raises the call option price. Conversely, it lowers the put option price.


## Option Greeks (cont'd)

- An increase in volatility raises the price of a call or put option, because it increases the expected value if the option is exercised.
- Options generally-but not always-become less valuable as time to expiration decreases, i.e., there is a time decay. There are exceptions, for example deep-in-the-money call options on an asset with high dividend yield and deep-in-the-money puts.
- A higher interest rate reduces the present value of the strike (to be paid by a call option holder), and thus increases the call price. The put option entitles the owner to receive the strike, whose present value is lower with a higher interest rate. Thus, a higher interest rate decreases the put price.
- A call entitles the holder to receive stock, but without dividends prior to expiration. Thus, the greater the dividend yield, the lower the call price. Conversely, a put option is more valuable when the dividend yield is greater.


## Option Greeks (cont'd)

- The Greeks are tools that let us to quantify these relationships:

| Input | Greek | Definition | Mnemonic |
| :---: | :---: | :---: | :---: |
| $S_{t}$ | $\Delta$ (Delta) | Measures the option price change when the stock price increases by $\$ 1$ |  |
| $S_{t}$ | 「 (Gamma) | Measures the change in $\Delta$ when the stock price increases by $\$ 1$ |  |
| $S_{t}$ | $\Omega$ (Elasticity) | Measures the percentage change in the option price when the stock price increases by $1 \%$ |  |
| $\sigma$ | Vega | Measures the option price change when there is an increase in volatility of $1 \%$ | vega $\leftrightarrow$ volatility |
| $t$ | $\theta$ (theta) | Measures the option price change when there is a decrease in the time to maturity (increase in calendar time) of 1 day | theta time |
| $r$ | $\rho$ (rho) | Measures the option price change when there is an increase in the interest rate of $1 \%$ (100 basis points) | $\underline{\text { rho }} \leftrightarrow \underline{r}$ |
| $\delta$ | $\Psi($ Psi) | Measures the option price change when there is an increase in the continuous dividend yield of $1 \%$ (100 basis points) |  |

## Option Greeks (cont'd)

- Let us come back to Table 1 and complete it with the proper signs of the Greeks:

| Input | Call Option |  | Put Option |  |
| :---: | :---: | :---: | :---: | :---: |
| $S_{t} \uparrow$ | $C_{t} \uparrow$ | $\Delta_{\text {Call }}>0$ | $P_{t} \downarrow$ | $\Delta_{\text {Put }}<0$ |
|  | $\Delta_{\text {Call }} \uparrow$ | $\Gamma_{\text {Call }}>0$ | $\Delta_{\text {Put }} \uparrow$ | $\Gamma_{\text {Put }}>0$ |
|  | $\Omega_{\text {call }} \geq 1$ |  | $\Omega_{\text {Put }} \leq 0$ |  |
| $\sigma \uparrow$ | $C_{t} \uparrow$ | Vegacall $>0$ | $P_{t} \uparrow$ | $V_{\text {ega }}^{\text {Put }}$ $>0$ |
| $t \uparrow$ | $C_{t}$ gen. $\downarrow$ | $\theta_{\text {Call }}$ gen. $<0$ | $P_{t}$ ambig. | $\theta_{\text {Put }}$ any |
| $r \uparrow$ | $C_{t} \uparrow$ | $\rho_{\text {Call }}>0$ | $P_{t} \downarrow$ | $\rho_{\text {Put }}<0$ |
| $\delta \uparrow$ | $C_{t} \downarrow$ | $\Psi_{\text {Call }}<0$ | $P_{t} \uparrow$ | $\Psi_{\text {Put }}>0$ |

## Option Greeks: Example

- The Greeks are mathematical derivatives of the option price formula with respect to the inputs.
- Suppose that the stock price is $S_{t}=\$ 41$, the strike price is $K=\$ 40$, volatility is $\sigma=0.3$, the risk-free rate is $r=0.08$, the time to expiration is $T-t=1$, and the dividend yield is $\delta=0$. The values for the Greeks are

| Input | Call Option |  | Put Option |  |
| :---: | :---: | :---: | :---: | :---: |
| $S_{t}$ | $\Delta_{\text {Call }}=0.691$ | $>0$ | Delta ${ }_{\text {Put }}=-0.309$ | <0 |
|  | $\Gamma_{\text {Call }}=0.029$ | $>0$ | $\Gamma_{\text {Put }}=0.029$ | $>0$ |
|  | $\Omega_{\text {Call }}=4.071$ | $\geq 1$ | $\Omega_{\text {Put }}=-4.389$ | $\leq 0$ |
| $\sigma$ | $V_{\text {ega }}$ all $=0.144$ | $>0$ | $V_{e g a p u t ~}=0.144$ | $>0$ |
| $t$ | $\theta_{\text {Call }}=-0.011$ | gen. $<0$ | $\theta_{\text {Put }}=-0.003$ | any |
| $r$ | $\rho_{\text {Call }}=0.214$ | $>0$ | $\rho_{\text {Put }}=-0.156$ | <0 |
| $\delta \uparrow$ | $\Psi_{\text {Call }}=-0.283$ | $<0$ | $\Psi_{\text {Put }}=0.127$ | $>0$ |

## Bloomberg: SPX 11/16/13 C1755 <INDEX> OV <GO>



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## Bloomberg: Greeks Convention



## Delta $(\Delta)$

Measures the change in the option price for a $\$ 1$ change in the stock price:


Gamma (Г)
Measures the change in delta when the stock price changes:


## Elasticity $(\Omega)$

Measures the percentage change in the option price relative to the percentage change in the stock price:


## Vega

Measures the change in the option price when volatility changes (divide by 100 for a change per percentage point):


## Theta $(\theta)$

Measures the change in the option price with respect to calendar time, $t$, holding fixed the maturity date $T$. To obtain per-day theta, divide by 365 .


## Rho ( $\rho$ )

Measures the change in the option price when the interest rate changes (divide by 100 for a change per percentage point, or by 10,000 for a change per basis point):


## Psi $(\Psi)$

Measures the change in the option price when the continuous dividend yield changes (divide by 100 for a change per percentage point):


## Gamma-Neutrality

- Gamma hedging is the construction of options positions that are hedged such that the total gamma of the position is zero.
- We cannot do this using just the stock, because the gamma of the stock is zero (the delta of a stock is constant and equal to 1 ).
- Hence, we must acquire another option in an amount that offsets the gamma of the written call.
- Let us go back to our previous example of delta-hedging. In addition to the 91 -day call, consider a 45 -strike 120 -day call.
- The ratio of the gamma of the two options is

$$
\begin{equation*}
\frac{\Gamma_{K=40, T-t=91}}{\Gamma_{K=45, T-t=120}}=\frac{0.0606}{0.0540}=1.1213 \tag{25}
\end{equation*}
$$

- Thus, we need to buy 1.1213 of the 45 -strike options for every 40-strike option we have sold.


## Gamma-Neutrality (cont'd)

- The Greeks resulting form this position are in the last column of the following table:

|  | 40-Strike Call | 45-Strike Call | Total Position |
| :---: | :---: | :---: | :---: |
| Price $(\$)$ | 3.395 | 1.707 | -1.481 |
| Delta $(\Delta)$ | 0.645 | 0.381 | -0.218 |
| Gamma $(\Gamma)$ | 0.061 | 0.054 | 0 |
| Theta $(\theta)$ | -0.018 | -0.014 | 0.002 |

- Since delta is -0.218 , we need to buy 21.8 shares of stock to be both delta- and gamma-hedged.


## Gamma-Neutrality (cont'd)

- We can compare the delta-hedged position with the delta- and gamma-hedged position. The delta-hedged position has the problem that large moves always cause losses.
- The delta- and gamma-hedged position loses less if there is a large move down, and can make money if the stock price increases.
- This can be seen from the following figure. It compares the 1-day holding period profit for delta-hedged position described earlier and delta- and gamma hedged position.


## Gamma-Neutrality (cont'd)

## $\rightarrow \quad$ Delta-hedged <br> $\rightarrow$ Delta- and gamma-hedged



## Gamma-Neutrality (cont'd)

- Delta-gamma hedging prevents the position from reacting to large changes in the underlying stock:



## No Hedge

## <HELP> for explanation.

|  | Action | S | 2) P | Position | ons |  | 3) Vi |  | W - | 4 Sett | tings | $\square$ | 99) Feed | ack |  | n | nario | io Anal | ysis |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| New | Portfolio | - | Unsa | saved Portt | rtfolio |  |  |  | < Rdd | osition > |  | USD |  |  | 25/13 | \# |  |  | Group |
|  | Positions | 33 H | Hedge | dge $\sqrt{33}$ | Scena | ario | Matrix |  | 34 Sc | nario Cha | art | 3.9. M | lulti-Asset | Scenar | 0 |  |  |  |  |
|  |  |  |  |  | Position |  | kt Px M |  | IVol | Cost | Total | Cost | Mkt Value | P\&L | Delta Not | tional | Delta | Gamma | Vega I |
| [-] P | Portfolio Su | ummary |  | $\square$ |  |  |  |  |  |  |  | 42 | 41 | -1 |  | 1,955 | 46 | 14 | 3.83 |
| -NFC | C US Equity |  |  | $\square$ |  |  |  |  |  |  |  | 42 | 41 | -1 |  | 1,955 | 46 | 14 | 3.83 |
|  | WFC US Equit |  |  | $\square$ | 0 | 4 | 42.86 | 1 |  | 42.86 |  | 0 | 0 | 0 |  | 0 | 0 | 0 | . 00 |
|  | WFC US 11/1 | 16/13 C4 | 43 | $\square$ | 1 | - | 0.41 m | m | 13.44 | 0.42 |  | 42 | 41 | -1 |  | 1,955 | 46 | 14 | 3.83 |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 53. | Scenario Ac | ctions |  | , |  |  |  |  | Scena | 0 Varying | g U/Px |  | - | Not | onal | P\&L. Fr |  | Cost |  |
|  | U/Px | Vol |  | Date | te Ra | ate |  |  | P\&L |  | P\&L \% |  | Delta |  | Gamma |  | Theta |  | Vega |
|  | Step - | Flat - |  |  | - Flat |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  | -/--/-- |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 11) | 39.00 | 0.00 |  | /25/13 | E 0.00 | 00 |  |  | -41.98 |  | -99.95 |  | . 06 |  | . 07 |  | -. 01 |  | 02 |
| 32) | 40.00 | 0.00 |  | 10/25/13 | - 0.00 | 00 |  |  | -41.7 |  | -99.28 |  | . 77 |  | . 64 |  | -. 06 |  | 2 |
| 33) | 41.00 | 0.00 |  | 10/25/13 $=$ | - 0.00 | 00 |  |  | -39.39 |  | 93.79 |  | 5.12 |  | 3.34 |  | -. 33 |  | . 99 |
| 34) | 42.00 | 0.00 |  | 10/25/13 | - 0.00 | 00 |  |  | -28.19 |  | 67.11 |  | 20.44 |  | 9.62 |  | -1.01 |  | 2.72 |
| 35) | 43.00 | 0.00 |  | 10/25/13 |  | 00 |  |  | 5.66 |  | 13.46 |  | 50.31 |  | 14.49 |  | -1.58 |  | 3.86 |
| 76) | 44.00 | 0.00 |  | 0/25/13 | = 0.00 | 00 |  |  | 71.29 |  | 69.74 |  | 80.55 |  | 10.57 |  | -1.18 |  | 2.69 |
| 77) | 45.00 | 0.00 |  | 10/25/13 $=$ | - 0.00 | 00 |  |  | 160.45 |  | 82.02 |  | 95.75 |  | 3.63 |  | -. 42 |  | 9 |
| 38) | 46.00 | 0.00 |  | 10/25/13 | - 0.00 | 00 |  |  | 258.53 |  | 15.55 |  | 99.51 |  | . 59 |  | -. 09 |  | 14 |
| 39) | 47.00 | 0.00 |  | 10/25/13 $=$ | \# 0.0 | . 00 |  |  | 358.35 |  | 53.21 |  | 99.97 |  | 05 |  | -. 03 |  | 01 |
| 919) Ex | ceptions |  |  | \$2) Beta Refe | ference |  |  |  |  |  |  |  |  |  | Zoom - |  |  | -1-1 | 100\% |

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## Delta-Hedged

## <HELP> for explanation.



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## Delta-Gamma-Hedged

## <HELP> for explanation.



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Japan $81332018900 \quad$ Singapore $6562121000 \quad$ U.S. $12123182000 \quad$ Copur ight 2013 Bloomberg Finance L. F,

## Calendar Spreads

- To protect against a stock price increase when selling a call, you can simultaneously buy a call option with the same strike and greater time to expiration.
- This purchased calendar spread exploits the fact that the written near-to-expiration option exhibits greater time decay than the purchased far-to-expiration option, and therefore is profitable if the stock price does not move.


## Calendar Spreads (cont'd)

- Suppose you sell a 40-strike call with 91 days to expiration and buy a 40-strike call with 1 year to expiration. Assume a stock price of $\$ 40$, $r=8 \%, \sigma=30 \%$, and $\delta=0$.
- The premiums are $\$ 2.78$ for the 91 -day call and $\$ 6.28$ for the 1 -year call.
- Theta is more negative for the 91-day call $(-0.0173)$ than for the 1 -year call ( -0.0104 ). Thus, if the stock price does not change over the course of 1 day, the position will make money since the written option loses more value than the purchased option.


## Calendar Spreads (cont'd)

- The profit diagram for this position for a holding period of 91 days is displayed below




## Appendix: Formulas for Option Greeks

- Delta $(\Delta)$ measures the change in the option price for a $\$ 1$ change in the stock price:

$$
\begin{align*}
\Delta_{C a l l} & =\frac{\partial C}{\partial S}=e^{-\delta(T-t)} N\left(d_{1}\right)  \tag{26}\\
\Delta_{\text {Put }} & =\frac{\partial P}{\partial S}=-e^{-\delta(T-t)} N\left(-d_{1}\right) \tag{27}
\end{align*}
$$

- Gamma ( $\Gamma$ ) measures the change in delta when the stock price changes:

$$
\begin{align*}
& \Gamma_{\text {Call }}=\frac{\partial^{2} C}{\partial S^{2}}=\frac{e^{-\delta(T-t)} N^{\prime}\left(d_{1}\right)}{S \sigma \sqrt{T-t}}  \tag{28}\\
& \Gamma_{\text {Put }}=\frac{\partial^{2} P}{\partial S^{2}}=\Gamma_{\text {Call }} \tag{29}
\end{align*}
$$

## Appendix: Formulas for Option Greeks (cont'd)

- Elasticity $(\Omega)$ measures the percentage change in the option price relative to the percentage change in the stock price:

$$
\begin{align*}
& \Omega_{\text {Call }}=\frac{S_{t} \Delta_{\text {Call }}}{C_{t}}  \tag{30}\\
& \Omega_{\text {Put }}=\frac{S_{t} \Delta_{\text {Put }}}{P_{t}} \tag{31}
\end{align*}
$$

- Vega measures the change in the option price when volatility changes (divide by 100 for a change per percentage point):

$$
\begin{align*}
& \text { Vega }_{\text {Call }}=\frac{\partial C}{\partial \sigma}=\text { Se }^{-\delta(T-t)} N^{\prime}\left(d_{1}\right) \sqrt{T-t}  \tag{32}\\
& \text { Vega }_{\text {Put }}=\frac{\partial P}{\partial \sigma}=\text { Vega }_{\text {Call }} \tag{33}
\end{align*}
$$

## Appendix: Formulas for Option Greeks (cont'd)

- Theta $(\theta)$ measures the change in the option price with respect to calendar time, $t$, holding fixed the maturity date $T$. To obtain per-day theta, divide by 365 .

$$
\begin{align*}
& \theta_{\text {Call }}=\frac{\partial C}{\partial t}=\delta S e^{-\delta(T-t)} N\left(d_{1}\right)-r K e^{-r(T-t)} N\left(d_{2}\right)-\frac{K e^{-r(T-t)} N^{\prime}\left(d_{2}\right) \sigma}{2 \sqrt{T-t}}  \tag{34}\\
& \theta_{\text {Put }}=\frac{\partial P}{\partial t}=\theta_{\text {Call }}+r K e^{-r(T-t)}-\delta S e^{-\delta(T-t)} \tag{35}
\end{align*}
$$

- Rho $(\rho)$ measures the change in the option price when the interest rate changes (divide by 100 for a change per percentage point, or by 10,000 for a change per basis point):

$$
\begin{align*}
\rho_{\text {Call }} & =\frac{\partial C}{\partial r}=(T-t) K e^{-r(T-t)} N\left(d_{2}\right)  \tag{36}\\
\rho_{\text {Put }} & =\frac{\partial P}{\partial r}=-(T-t) K e^{-r(T-t)} N\left(-d_{2}\right) \tag{37}
\end{align*}
$$

## Appendix: Formulas for Option Greeks (cont'd)

- Psi $(\Psi)$ Measures the change in the option price when the continuous dividend yield changes (divide by 100 for a change per percentage point):

$$
\begin{align*}
\Psi_{\text {Call }} & =\frac{\partial C}{\partial \delta}=-(T-t) S e^{-\delta(T-t)} N\left(d_{1}\right)  \tag{38}\\
\Psi_{\text {Put }} & =\frac{\partial P}{\partial \delta}=(T-t) S e^{-\delta(T-t)} N\left(-d_{1}\right) \tag{39}
\end{align*}
$$

Financial Derivatives FINE 448
5. Hedging and Pricing Volatility

Daniel Andrei

## (4. McGill

Winter 2021

## Outline

I Volatility ..... 3

- Measurement and Behavior of Volatility ..... 3
- Implied Volatility ..... 7
- The CBOE Volatility Index (VIX) ..... 13
- Volatility Trading ..... 18


## The Use of Volatility in Finance

- for forecasting return
- for the pricing of derivatives
- for asset allocation (trade-off between return and risk)
- for risk management (evaluation of the risk of a portfolio)


## Unconditional vs Conditional Volatility

- Unconditional volatility is estimated as the sample standard deviation

$$
\begin{equation*}
\sigma=\sqrt{\frac{1}{T} \sum_{t=1}^{T}\left(r_{t}-\mu\right)^{2}} \tag{1}
\end{equation*}
$$

where $r_{t}$ is the log return on period $t$ and $\mu$ is the sample mean over $T$ periods

- However, volatility is actually not constant through time. Therefore, conditional volatility $\sigma_{t}$ is a more relevant measure of risk at time $t$.


## Volatility Clustering



## GARCH $(1,1)$ volatility for the S\&P 500 (weekly data)




## Implied Volatility

- Volatility is unobservable
- Choosing a volatility to use in pricing an option is difficult but important
- Using history of returns is not the best approach, because history is not a reliable guide to the future.
- We can invert Black-Scholes formula to obtain implied volatility
- We cannot use implied volatility to assess whether an option price is correct, but implied volatility does tell us the market's assessment of volatility


## Implied Volatility (cont'd)

- Example: Let $S=\$ 100, K=\$ 90, r=8 \%, \delta=5 \%$, and $T=1$. The market option price for a call option is $\$ 18.25$. What is the volatility that gives this option price?
- We must invert the following formula

$$
\$ 18.25=\text { BSCall }(100,90, \sigma, 0.08,1,0.05)
$$

- We use the implied volatility function:


We find that setting $\sigma=31.73 \%$ gives us a call price of $\$ 18.25$

## Implied Volatility (cont'd)

- There is a systematic pattern of implied volatility across strike prices, called volatility skew
- The volatility skew is not related to whether an option is a put or a call, but rather to differences in the strike price and time to expiration
- Explaining these patterns is a challenge for option pricing theory


## Bloomberg: SPX <INDEX> SKEW <GO>

## <HELP> for explanation.

## Enter all values and hit <Go>




Australia 61297778600 Brazil 551130484500 Europe 442073307500 Germany 496992041210 Hong Kong 85229776000 Japan 81332018900 Singapore 6562121000 U.S. 12123182000 Copuright 2013 Bloomberg Finance L.P SN 184792 PDT GMT-7:00 H698-797-0 22-0ct-2013 14:12:47

## Bloomberg: WFC US <EQUITY> SKEW <GO>

## <HELP> for explanation.

Enter all values and hit <Go>


- 1) wFC U5: Market Nov '13 TD Call (Price: 42.94)
- 2) WFC US: Narket Aer '14 TD Cell (Price: 42.94)


## Using Implied Volatility

- Implied volatility is important for a number of reasons
- If you need to price an option for which you cannot observe a market price, you can use implied volatility to generate a price consistent with the price of traded options
- Implied volatility is often used as a quick way to describe the level of option prices on a given underlying asset. Option prices are quoted sometimes in terms of volatility, rather than as a dollar price
- Volatility skew provides a measure of how well option pricing models work
- Just as stock markets provide information about stock prices and permit trading stocks, option markets provide information about volatility, and, in effect, permit the trading of volatility.


## The CBOE Volatility Index (VIX)

- In 1993, the Chicago Board Options Exchange (CBOE) introduced the CBOE Volatility Index, VIX, which was originally designed to measure the market's expectation of 30 -day volatility implied by at-the-money S\&P 100 Index option prices
- Ten years later in 2003, CBOE and Goldman Sachs updated the VIX to reflect a new way to measure expected volatility. The new VIX is based on the S\&P 500 Index, and estimates expected volatility by averaging the weighted prices of puts and calls over a wide range of strike prices.


## The CBOE Volatility Index (VIX)

- The CBOE utilizes a wide variety of strike prices for SPX puts and calls to calculate the VIX
- VIX provides important information about investor sentiment. Since volatility often signifies financial turmoil, the VIX is often referrred to as the investor fear gauge.
- Investors can use VIX options and VIX futures to hedge their portfolios. The VIX is a good hedging tool because it has a strong negative correlation to the S\&P 500


## The CBOE Volatility Index (VIX)



## Bloomberg: VIX <INDEX> DES <GO>

```
VIX T13.59 +.92 rev\mur
At 12:07 d 0 12.99 H 13.79 L 12.93 Prev 12.67
```

VIX Index

## 99) Feedback

## CBOE SPX VOLATILITY INDX

The Chicago Board Options Exchange Volatility Index reflects a market estimate of future volatility, based on the weighted average of the implied volatilities for a wide range of strikes 1st \& 2nd month expirations are used until 8 days from expiration, then the 2nd and 3rd are used. [BBGID BBG000JW9B77]
3) Price Chart (GP)


Prices
5) Intraday Chart (GIP) Last
6) Bar Chart (GP0) 52 Wk High

52 Wk Low

06:30-13:15
USD
N.A.

| 4 Return Analysis (TRA) | \% Chg | Annual |  |
| :--- | ---: | ---: | ---: |
| 1 Day | 12.67 | +7.18 | + Lge |
| 5 Days | 13.75 | -1.24 | -47.73 |
| MTD | 13.75 | -1.24 | -47.73 |
| QTD | 16.60 | -18.19 | -85.47 |
| YTD | 18.02 | -24.64 | -28.25 |
| 1 Month | 19.41 | -30.04 | -98.51 |
| 3 Months | 12.98 | +4.62 | +19.64 |
| 6 Months | 12.83 | +5.85 | +11.93 |
| 1 Year | 19.08 | -28.83 | -28.83 |
| 2 Years | 29.85 | -54.51 | -32.51 |
| 5 Years | 56.10 | -75.79 | -24.69 |
| Qtr 3:12 | 17.08 | -7.90 | -28.39 |
| Qtr 4:12 | 15.73 | +14.56 | +73.53 |
| Qtr 1:13 | 18.02 | -29.52 | -75.80 |
| Qtr 2:13 | 12.70 | +32.76 | +215.54 |
| Qtr 3:13 | 16.86 | -1.54 | -6.11 |

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Japan 81332018900 Singapore 6562121000 U.S. 12123182000 Copuright 2013 Bloomberg Finance L.P.

## Computing the VIX

- See additional note (to be discussed in class)


## Volatility Trading

- Just as stock investors think they know something about the direction of the stock market, or bond investors think they can foresee the probable direction of interest rates, so you may think you have insight into the level of future volatility
- What do you do if you simply want exposure to a stock's volatility?
- Stock options are impure: they provide exposure to both the direction of the stock price and its volatility


## Volatility Trading: The Traditional Way

- Buy/sell straddles or strangles. Easy to implement, but has drawbacks:
- Straddles and strangles yield a non-null delta once the stock price moves away from the initial ATM strike price
- Prices need to move sharply in the case of a buy-and-hold strategy
- Option delta-hedging (avoiding sensitivity to asset price). Drawbacks:
- The P\&L generated by delta hedging an option is a function of numerous sources of risk: variance risk, volatility path dependency risk, model risk, liquidity risk, dividends risk.
- Variance risk sometimes accounts only for $50 \%$ of the total P\&L.


## Advantages of Variance and Volatility Swaps

- The easy way to trade volatility is to use volatility swaps, sometimes called realized volatility forward contracts
- These products provide pure exposure to volatility (and only to volatility)
- No need to delta hedge $\Rightarrow$ allows for buy-and-hold variance strategy
- OTC products but standardized contracts with maturity similar to listed options (April 14, June 14, ...)
- Strong liquidity thanks to several investment banks providing live prices
- Either long or short positions
- On indices as well as single stocks


## Variance Swaps

- A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

$$
\begin{equation*}
\left[\widehat{V}^{2}-F_{0, T}\left(V^{2}\right)\right] \times N \tag{2}
\end{equation*}
$$

where

- $\widehat{V}^{2}$ is the realized stock variance over the life of the contract
- $F_{0, T}\left(V^{2}\right)$ is the forward price for variance
- $N$ is the notional amount of the swap in dollars per annualized volatility point squared


## Example of a Variance Swap

- Three-month S\&P 500 variance futures (Chicago Futures Exchange)
- The payoff is based on the annualized sum of squared, continuously compounded daily returns over a 3-month period, $\widehat{V}^{2}$
- The measured price is quoted as $\widehat{V}^{2} \times 10,000$, and by definition a one-unit change in this number (a variance point) is worth \$50
- The payoff at expiration is

$$
\begin{equation*}
\$ 50 \times\left[10,000 \times 252 \times \sum_{i=1}^{n_{a}-1} \frac{\epsilon_{i}^{2}}{n_{e}-1}-F_{0, T}\left(V^{2}\right)\right] \tag{3}
\end{equation*}
$$

where

- $\epsilon_{i}=$ continuously compounded return on day $i$
- $n_{a}=$ actual number of S\&P prices used to compute the variance
- $n_{e}=$ expected number of trading days from 0 to $T$


## Volatility Swaps

- A volatility swap is like a variance swap except that it pays based on volatility rather than variance:

$$
\begin{equation*}
\left[\widehat{V}-F_{0, T}(V)\right] \times N \tag{4}
\end{equation*}
$$

where $F_{0, T}(V)$ is the forward price of volatility.

- Example: the volatility futures contract (Chicago Futures Exchange):

$$
\begin{equation*}
1000 \times\left[\mathrm{VIX}_{T}-F_{0, T}(V)\right] \tag{5}
\end{equation*}
$$

- Variance swaps are easier to value than volatility swaps (the variance can be replicated using a portfolio of options; there is no simple replication strategy for synthesizing a volatility swap)
- Delta-hedging dealers use variance swaps to hedge risk in realized variance


## Bloomberg: UXZ3 <INDEX> DES < GO>



## Bloomberg: VIX UX 11/20/13 C14 <INDEX> OV <GO>



## Bloomberg: VIX UX 11/20/13 P14 <INDEX> OV <GO>



Australia 61297778600 Brazil 551130484500 Europe 442073307500 Germany 496992041210 Hong Kong 85229776000 Japan 81332018900 Singapore 6562121000 U.S. 12123182000 Copuright 2013 Bloomberg Finance L.P. SN 264328 PST GMT-8:00 H706-3196-0 07-Now-2013 12:40:39



# Financial Derivatives FINE 448, Winter 2021 

# Computing the VIX 

Daniel Andrei, McGill

## 1 The Log Contract

Neuberger (1994) pointed out that a forward contract that pays

$$
\begin{equation*}
\ln \left(\frac{S_{T}}{K}\right) \tag{1}
\end{equation*}
$$

could be used to hedge or speculate on variance. The value of this contract at time $t=0$ for a non-dividend paying asset is

$$
\begin{equation*}
L_{0}=e^{-r T}\left[\ln \left(\frac{S_{0}}{K}\right)+\left(r-\frac{1}{2} \sigma^{2}\right) T\right] \tag{2}
\end{equation*}
$$

Fix the strike to the forward price with maturity $T$, that is, $K=F_{0, T}=e^{r T} S_{0}$. Replacing $K$ in equation (2) yields

$$
\begin{align*}
L_{0} & =e^{-r T}\left[\ln \left(\frac{S_{0}}{e^{r T} S_{0}}\right)+\left(r-\frac{1}{2} \sigma^{2}\right) T\right]  \tag{3}\\
& =e^{-r T}\left[\ln \left(e^{-r T}\right)+\left(r-\frac{1}{2} \sigma^{2}\right) T\right]  \tag{4}\\
& =e^{-r T}\left[-r T+r T-\frac{1}{2} \sigma^{2} T\right]  \tag{5}\\
& =-\frac{1}{2} e^{-r T} T \sigma^{2} \tag{6}
\end{align*}
$$

Solving for $\sigma^{2}$ yields

$$
\begin{equation*}
\sigma^{2}=\frac{2 e^{r T}}{T}\left(-L_{0}\right) \tag{7}
\end{equation*}
$$

Equation (7) demonstrates the connection between the price of the log contract and variance. Thus, buying or selling such a log contract would allow us to hedge or speculate on variance. Yet, as of today, there is no exchange-traded log contract in existence. There is a trick, however, for pricing the log contract using other instruments.

Carr and Madan (1998) and Demeterfi, Derman, Kamal, and Zou (1999) independently showed that it is possible to use a portfolio of options to replicate the payoff on the log contract. ${ }^{1}$ More precisely, one can write

$$
\begin{equation*}
-L_{0}=\int_{0}^{F_{0, T}} \frac{1}{K^{2}} \operatorname{Put}(K) d K+\int_{F_{0}, T}^{\infty} \frac{1}{K^{2}} \operatorname{Call}(K) d K \tag{8}
\end{equation*}
$$

which can be replaced in (7):

$$
\begin{equation*}
\sigma^{2}=\frac{2 e^{r T}}{T}\left[\int_{0}^{F_{0, T}} \frac{1}{K^{2}} \operatorname{Put}(K) d K+\int_{F_{0, T}}^{\infty} \frac{1}{K^{2}} \operatorname{Call}(K) d K\right] \tag{9}
\end{equation*}
$$

Remarkably, this formula gives us an estimate of stock return variance that we compute using the observed prices of out-of-the-money puts and calls! ("Out-of-themoney" here is with respect to the forward price rather than the current stock price.)

## 2 Computing the VIX

We can now explain the formula used to compute the VIX. The calculation is based on equation (9). In practice, option strike prices are discrete and there may be no option for which the strike price equals the index forward price. The actual formula used by the CBOE is a discrete approximation to equation (9):

$$
\begin{equation*}
\sigma^{2}=\frac{2 e^{r T}}{T} \sum_{K_{i} \leq K_{0}} \frac{\Delta K_{i}}{K_{i}^{2}} \operatorname{Put}\left(K_{i}\right)+\frac{2 e^{r T}}{T} \sum_{K_{i}>K_{0}} \frac{\Delta K_{i}}{K_{i}^{2}} \operatorname{Call}\left(K_{i}\right)-\frac{1}{T}\left(\frac{F_{0, T}}{K_{0}}-1\right)^{2} \tag{10}
\end{equation*}
$$

where $K_{0}$ is the first strike below the forward price and $\Delta K_{i}=\left(K_{i+1}-K_{i-1}\right) / 2$. The last term is a correction for the fact that there may be no option with a strike equal to the forward price.

At the upper and lower edges of any given strip of options, $\Delta K_{i}$ is simply the absolute value of the difference between $K_{i}$ and the adjacent strike price.

[^0]
## Example

Today's date and time is February $19^{\text {th }} 2016,15: 00: 00$. The interest rate is $\mathrm{r}=0.0028$. Download the option data provided in the Excel file "VIXdata.xlsx", which contains two worksheets:

- Worksheet "Short" contains options that expire on Mar 182016 at 15:00 (i.e., "near-term" options)
- Worksheet "Long" contains options that expire on Mar 252016 at 15:00 (i.e., "next-term" options)
Follow the step-by-step calculation in the document "vixwhite.pdf" and answer the questions below.

1. What is the time to expiration of near-term options, $T_{1}$ (in years)? What is the time to expiration of next-term options, $T_{2}$ (in years)? Note: a year has 525,600 minutes.

Answer: $T_{1}=0.0767, T_{2}=0.0959$.
2. For both near- and next-term options, determine the forward SPX levels, $F_{1}$ and $F_{2}$, by identifying the strike price at which the absolute difference between call and put prices is smallest. Next, determine $K_{0,1}$ and $K_{0,2}$ (the strike prices immediately below the forward index levels, $F_{1}$ and $F_{2}$ ) for the near- and nextterm options.

Answer: $F_{1}=1914.5, F_{2}=1905.7 . K_{0,1}=1910, K_{0,2}=1905$.
3. Calculate the variance for both near- and next-term options, $\sigma_{1}^{2}$ and $\sigma_{2}^{2}$.

Answer: $\sigma_{1}^{2}=0.0414, \sigma_{2}^{2}=0.0408$.
4. Calculate the 30-day weighted average of $\sigma_{1}^{2}$ and $\sigma_{2}^{2}$. Then take the square root of that value and multiply by 100 to get the VIX value.

Answer: $V I X=20.2946$.

## References

Carr, P. and D. Madan (1998). Towards a theory of volatility trading. Volatility: New estimation techniques for pricing derivatives 29, 417-427.

Demeterfi, K., E. Derman, M. Kamal, and J. Zou (1999). A guide to volatility and variance swaps. The Journal of Derivatives 6(4), 9-32.

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# Financial Derivatives FINE 448 

6. Black-Scholes: Practical Uses

Daniel Andrei

## 界 McGill

Winter 2021

## Outline

I Practical Uses of the Black-Scholes Model 3

- Real Options Revisited 3
- The Option to Abandon 8
- Portfolio Insurance 11
- Collars in Acquisitions: Valuing an Offer15


## Real Options Revisited: Evaluating Projects with an Infinite Investment Horizon

- Consider the investment under uncertainty problem from the binomial option pricing section.
- A project requires an initial investment of $\$ 100$. Thus, $K=100$.
- The project is expected to generate a perpetual cash flow stream, with a first cash flow $\$ 18$ in one year, expected to grow at $3 \%$ annually. Assume a discount rate of $15 \%$.

$$
\Rightarrow\left\{\begin{array}{l}
\text { Perpetual growing annuity } \Rightarrow P V=\frac{\$ 18}{0.15-0.03}=\$ 150  \tag{1}\\
\text { Static NPV }=\$ 150-\$ 100=\$ 50 \\
\text { Cont. compounded div. yield } \delta=\ln \left(\frac{\$ 18}{\$ 150}+1\right)=0.1133
\end{array}\right.
$$

- The cont. compounded risk-free rate is $r=6.766 \%$. The cash flows of the project are normally distributed with a volatility of $\sigma=50 \%$.


## Real Options Revisited: Evaluating Projects with an Infinite Investment Horizon (cont'd)

- The above example assumes that we must start the project by year 2 .
- Suppose instead that the project can be started at any time and then will live forever.
- The project is then a perpetual call option


## Valuing Perpetual Options

- Calls and puts that never expire ar known as perpetual options.
- Perpetual American options always have the same time to expiration, namely infinity.
- Because time to expiration is constant, the option exercise problem will look the same today, tomorrow, and forever.
- Thus, the price at which it is optimal to exercise the option is constant.
- The optimal exercise strategy entails picking the exercise barrier that maximizes the value of the option, and then exercising the option the first time the stock price reaches that barrier.


## Valuing Perpetual Options (cont'd)

- First, define $h_{1}$ and $h_{2}$ :

$$
\begin{align*}
& h_{1}=\frac{1}{2}-\frac{r-\delta}{\sigma^{2}}+\sqrt{\left(\frac{r-\delta}{\sigma^{2}}-\frac{1}{2}\right)^{2}+\frac{2 r}{\sigma^{2}}}  \tag{2}\\
& h_{2}=\frac{1}{2}-\frac{r-\delta}{\sigma^{2}}-\sqrt{\left(\frac{r-\delta}{\sigma^{2}}-\frac{1}{2}\right)^{2}+\frac{2 r}{\sigma^{2}}} \tag{3}
\end{align*}
$$

- Perpetual American Call:

$$
\begin{equation*}
C_{\text {perpetual }}=\left(H_{c}-K\right)\left(\frac{S}{H_{c}}\right)^{h_{1}}, \text { where } H_{c}=K \frac{h_{1}}{h_{1}-1} \tag{4}
\end{equation*}
$$

- Perpetual American Put:

$$
\begin{equation*}
P_{\text {perpetual }}=\left(K-H_{p}\right)\left(\frac{S}{H_{p}}\right)^{h_{2}}, \text { where } H_{p}=K \frac{h_{2}}{h_{2}-1} \tag{5}
\end{equation*}
$$

## Real Options Revisited: Evaluating Projects with an Infinite Investment Horizon (cont'd)

- Using continuously compounded imputs, we compute:

$$
\begin{equation*}
\text { CallPerpetual }(\$ 150, \$ 100,0.5,0.06766,0.1133)=\{\$ 63.4, \$ 245.7\} \tag{6}
\end{equation*}
$$

- When the project value is $\$ 150$, the option value is $\$ 63.4$ and the optimal investment trigger is $\$ 245.7$.
- In other words, we invest when the project is worth $\$ 245.7$, more than twice the investment cost.
- If we invest immediately, the project is worth $\$ 50$.
- The ability to wait increases that value by $\$ 13.4$.


## The Option to Abandon

- Firms worry that new projects will not pay off
- Having the option to abandon a project that does not pay off can be valuable
- The option to abandon takes on the characteristics of a put option:
- $V$ is the remaining value on a project if it continues to the end of its life
- $L$ is the liquidation (abandonment) value
- Payoff from owning an abandonment option:

$$
\text { Payoff }= \begin{cases}0 & \text { if } V>L \\ L-V & \text { if } V \leq L\end{cases}
$$

- Having a option to abandon a project can make otherwise unacceptable projects acceptable


## The Option to Abandon: Example

- Assume that Blue Star Aircraft is interested in building a small passenger plane and that it approaches Boeing with a proposal for a joint venture. Each firm will invest $\$ 500$ million in the joint venture. The investment is expected to have an infinite lifespan.
- Boeing works through a traditional investment analysis and concludes that their share of the present value of expected cash flows would be only $\$ 470$ million. The net present value of the project would therefore be negative (NPV $=-\$ 30$ million) and Boeing rejects this joint venture.
- On rejection of the joint venture, Blue Star approaches Boeing with a sweetener, offering the option to buy out Boeing's share of the joint venture at any time into the future starting from today, for $\$ 300$ million. Although this is much less than what Boeing will invest initially, it puts a floor on the losses and thus gives Boeing an abandonment option (that is, a perpetual put option).

1. Value the abandonment option. Use a standard deviation of $\sigma=0.3$ and a dividend yield of $\delta=1 / 30$. The risk-free rate is $5 \%$.
Answer: The abandonment option is a perpetual put with strike $\$ 300$ million, with value

$$
\begin{equation*}
\text { PUTPERPETUAL }(470,300,0.3,5 \%, 1 / 30)=\$ 61.97 \text { million } . \tag{7}
\end{equation*}
$$

2. Given this additional option, should Boeing enter this joint venture?

Answer: Boeing should enter into this joint venture, because the value is greater than the negative net present value of the investment (NPV $=-\$ 30$ million). The fact that the option to abandon has value provides a rationale for Boeing to build the operating flexibility to terminate the project if it does not measure up to expectations.
3. Assume that Boeing enters this joint venture. According to calculations made at point 1, when should Boeing "abandon" the project and sell back its share?
Answer: The barrier of the perpetual put is $\$ 131.96$ million. Boeing should abandon the project when the present value of expected cash flows touches this barrier.

## Portfolio Insurance

- A portfolio manager is often interested in acquiring a put option on his or her portfolio
- The option can be created synthetically
- This involves maintaining a position in the underlying asset so that the delta of the position is equal to the delta of the required option
- There are two reasons why it might be more attractive to create the required put option synthetically than to buy it in the market:
- Option markets do not always have the liquidity to absorb the trades required by managers of large funds
- Fund managers often require strike prices and exercise dates that are different from those available in exchange-traded option markets


## Portfolio Insurance

## Example

A portfolio is worth $\$ 90$ million. To protect against market downturns the managers of the portfolio require a 6-month European put option on the portfolio with a strike of $\$ 87$ million.

The risk-free rate is $9 \%$ per annum, the dividend yield is $3 \%$ per annum, and the volatility of the portfolio is $25 \%$ per annum. The S\&P 500 stands at 900 .

The portfolio is considered to mimic the S\&P 500 fairly closely.
Create the required option synthetically.

## Portfolio Insurance

- In this case, $S_{0}=90$ million, $K=87$ million, $r=0.09, \delta=0.03$, $\sigma=0.25$, and $T=0.5$
- The delta of the put option is

$$
\begin{equation*}
-e^{-\delta T} N\left(-d_{1}\right)=-0.3215 \tag{8}
\end{equation*}
$$

- This shows that $32.15 \%$ of the portfolio ( $\$ 28.94$ million) should be sold initially and invested in risk-free assets to match the delta of the required option.
- The amount of the portfolio should be monitored frequently:
- If the portfolio reduces to $\$ 88$ million after 1 day, the delta of the required option changes to -0.3681 and further $4.66 \%$ of the original portfolio ( $\$ 4.10$ million) should be sold and invested in the risk-free asset
- If the portfolio increases to $\$ 92$ million, delta changes to -0.2787 and $4.29 \%$ ( $\$ 3.94$ million) should be repurchased
- Calculations are more involved if the portfolio's beta $(\beta)$ is not 1 .


## Example: Option-Based Portfolio Insurance (OBPI)

 strategy over 298 trading days, for an initial investment of $\$ 100$, with a floor of $\$ 95$.Stock Price and Position Value


## Collars in Acquisitions: Valuing an Offer

## The Northrop Grumman-TRW merger

In July 2002, Northrop Grummann and TRW agreed that Northrop would pay $\$ 7.8$ billion for TRW. The number of Northrop Grumman shares to be exchanged for each TRW share is

$$
\begin{aligned}
0.5357 \text { shares } & \text { if } S_{N G} \leq \$ 112 \\
\$ 60 / S_{N G} \text { shares } & \text { if } \$ 112<S_{N G}<\$ 138 \\
0.4348 \text { shares } & \text { if } S_{N G} \geq \$ 138
\end{aligned}
$$

where $S_{N G}$ is the average Northrop Grumman price over the 5 days preceding the close of the merger.

Suppose that TRW shareholders were certain the merger would occur at time $T=5 / 12$. Assume a risk-free rate of $1.5 \%$, and the volatility Northrop Grumman shares is $36 \%$. Northrop Grumman pays no dividends. The closing price of Northrop Grumman is $\$ 120$.

How would TRW shareholders value the Northrop offer?

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- The offer is equivalent to:

1. Buying 0.5357 shares of Northrop Grumman
2. Selling 0.5357 112-strike calls (the Black-Scholes price is $\$ 15.6 / \mathrm{call}$ )
3. Buying 0.4348138 -strike calls (the Black-Scholes price is $\$ 5.22 / \mathrm{call}$ )

- To understand this, plot the value of the Northrop Grumman offer for one TRW share, as a function of the Northrop Grumman share price (you should obtain a floating collar offer-see page 501 in McDonald and next slide)
- The value of TRW shares would then be

$$
\begin{equation*}
0.5357 \times 120-0.5357 \times 15.6+0.4348 \times 5.22=\$ 58.2 \tag{9}
\end{equation*}
$$

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## FIGURE 16.8

Value of Northrop Grumman offer for TRW at closing of the merger and with $4 \frac{1}{2}$ morths until closing.


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- The theoretical value of a TRW share under the terms of the offer is greater than the market price of a TRW share
- This is what we would expect to see, since in order to induce the target company to accept an offer, the acquirer generally has to offer a price greater than the perceived value of the target as a stand-alone company
- The difference between the two values declines toward zero as the merger is likely to take place or diverges if the merger is cancelled for some reason
- Risk arbitrageurs take positions in the two stocks in order to speculate on the success or failure of the merger
- Mitchell and Pulvino (Characteristics of Risk and Return in Risk Arbitrage, Journal of Finance, 2001) examine the historical returns earned by risk arbitrageurs


Figure 1. This figure plots the median arbitrage spread versus time until deal resolution. The arbitrage spread is defined to be the offer price minus the target price divided by the target price. For failed deals, the deal resolution date is defined as the date of the merger termination announcement. For successful deals, the resolution date is the consummation date.

## Other Uses of the Black-Scholes Model

- Financial Engineering:
- Convertible Bonds \& Reverse Convertible Bonds
- Tranched Payoffs
- Corporate Applications:
- Warrants
- Compensation Options
- Other Uses of Real Options:
- Commodity Extraction with Shutdown and Restart Options
- ... see Chapters 15,16 , and 17 in textbook.


[^0]:    ${ }^{1}$ Emanuel Derman, a physicist turned financial engineer, is the author of the well-known book "My life as a quant" (Derman, 2004).

