McGill Desautels Faculty of Management
Daniel Andrei, Financial Derivatives FINE 448 (BCom), Winter 2020
Midterm A (Morning Section)
February 2020

Date: $\qquad$

Your Name: $\qquad$

Section (Morning/Afternoon): $\qquad$

Your Signature: ${ }^{1}$

- This exam is open book, open notes. You can use a calculator, but be sure to show or explain your work.
- You are not allowed to communicate with anyone (verbally, in writing, or electronically), during the exam period. I will not answer questions.
- You may present calculations in non-reduced form (e.g., as " $e^{0.095}-1$ ").
- If you are stuck on something, make an assumption, tell me what it is, and do the best you can. I give partial credit if you provide enough correct information.


## TIME LIMIT: 1 hour

TOTAL POINTS: 100

[^0]
## 1 (20 points) Exotic Option Valuation

You have been recently hired by The Arbitrage Whale, a start-up company based in Tadoussac, Québec. Patrick Augustin, the CEO of the company, needs your help for pricing an exotic option.

Patrick uses a binomial tree with 25 periods to valuate this option. In the table below, the column "Node" denotes the number of "up" movements on the tree. The column " $q$ " denotes the probability of reaching each one of the 26 terminal nodes. Finally, the payoff of this exotic option at each terminal node equals exactly the probability of reaching that node, as you can see in the column "Payoff."

The maturity of the option is $T=1$. The interest rate is $r=0 \%$. Patrick has spent some time to compute the variance of column " $q$ " for you: $\operatorname{Var}[q]=0.00284097$, then took some time off to go whale watching.
Note: There are two ways to solve this question, one that is tedious and timeconsuming and one that takes only 3-5 minutes and requires very little calculus.

| Node | $q=\binom{n}{k}\left(p^{*}\right)^{k}\left(1-p^{*}\right)^{n-k}$ | Payoff $=q$ |
| :---: | :---: | :---: |
| $k=25$ | 0.00000001 | 0.00000001 |
| $k=24$ | 0.00000036 | 0.00000036 |
| $k=23$ | 0.00000470 | 0.00000470 |
| $k=22$ | 0.00003833 | 0.00003833 |
| $k=21$ | 0.00022385 | 0.00022385 |
| $k=20$ | 0.00099832 | 0.00099832 |
| $k=19$ | 0.00353352 | 0.00353352 |
| $k=18$ | 0.01018408 | 0.01018408 |
| $k=17$ | 0.02433112 | 0.02433112 |
| $k=16$ | 0.04880072 | 0.04880072 |
| $k=15$ | 0.08290942 | 0.08290942 |
| $k=14$ | 0.12004943 | 0.12004943 |
| $k=13$ | 0.14871836 | 0.14871836 |
| $k=12$ | 0.15791459 | 0.15791459 |
| $k=11$ | 0.14372527 | 0.14372527 |
| $k=10$ | 0.11191601 | 0.11191601 |
| $k=9$ | 0.07427282 | 0.07427282 |
| $k=8$ | 0.04175237 | 0.04175237 |
| $k=7$ | 0.01970408 | 0.01970408 |
| $k=6$ | 0.00770829 | 0.00770829 |
| $k=5$ | 0.00245548 | 0.00245548 |
| $k=4$ | 0.00062079 | 0.00062079 |
| $k=3$ | 0.00011985 | 0.00011985 |
| $k=2$ | 0.00001659 | 0.00001659 |
| $k=1$ | 0.00000146 | 0.00000146 |
| $k=0$ | 0.00000006 | 0.00000006 |

a. (10 points) What is the average of column " $q$ "?

$$
\text { Average of column " } q \text { " }
$$

b. (10 points) Compute the price of this exotic option today.

> Price of the exotic option today

## 1 Solution

a. The average is

$$
\begin{equation*}
\mathbb{E}[q]=\frac{1}{26} \sum_{k=0}^{25} q_{k}=\frac{1}{26}=0.03846154 \tag{1}
\end{equation*}
$$

(we know that the sum of probabilities is 1 )
b. The interest rate is zero, and we also know that all the payoffs are the same as the probabilities, so we just have to compute the sum of $q^{2}$, or $26 \times \mathbb{E}\left[q^{2}\right]$. We further know that

$$
\begin{equation*}
\operatorname{Var}[q]=\mathbb{E}\left[q^{2}\right]-\mathbb{E}[q]^{2} \tag{2}
\end{equation*}
$$

and therefore

$$
\begin{align*}
26 \times \mathbb{E}\left[q^{2}\right] & =26 \times\left(\operatorname{Var}[q]+\mathbb{E}[q]^{2}\right)  \tag{3}\\
& =26 \times\left(0.00284097+0.03846154^{2}\right)  \tag{4}\\
& =0.11232673 \tag{5}
\end{align*}
$$

## 2 (40 points) Option to reduce capacity

McGill Desautels is considering starting a new Master of Retail Management. The Master will generate cash flows two years from now, as follows:

- The cash flows will be $\$ 240$ million following two good years
- The cash flows will be $\$ 150$ million following one good year and one bad year
- The cash flows will be $\$ 100$ million following two bad years

The initial cost of starting the Master of Retail Management is $\$ 140$ million. Consider a two-period binomial tree with maturity $T=2$ years, and assume a riskfree rate of $10 \%$ per annum (continuously compounded). The risk-neutral probability of having a good year is $p^{*}=0.54$.
a. (10 points) What is the NPV of this project today (at time 0)? Should McGill Desautels start this new Master?
NPV at time 0

After one year, McGill Desautels has the option to reduce by half the intake of students (which will reduce by half future cash flows), and to sell the unused space to the Société Québécoise du Cannabis (SQDC) for $\$ 75$ million.
b. (10 points) Assume that one good year has passed and we are now at time 1. Should McGill Desautels reduce the intake of students and sell the unused space to the SQDC?
c. (10 points) Assume that one bad year has passed and we are now at time 1. Should McGill Desautels reduce the intake of students and sell the unused space to the SQDC?
d. (10 points) Compute the new NPV of this project today (at time 0), under the scenario that the option to reduce capacity is not ignored. Given this new NPV, should McGill Desautels start this new Master?

$$
\text { New NPV at time } 0
$$

## 2 Solution

a. The NPV of the project today is

$$
\begin{align*}
N P V & =e^{-0.1 \times 2}\left(240 \times\left(p^{*}\right)^{2}+150 \times 2 p^{*}\left(1-p^{*}\right)+100 \times\left(1-p^{*}\right)^{2}\right)-140  \tag{6}\\
& =e^{-0.1 \times 2}(240 \times 0.2916+150 \times 0.4968+100 \times 0.2116)-140  \tag{7}\\
& =e^{-0.1 \times 2} 165.66-140  \tag{8}\\
& =135.63-140  \tag{9}\\
& =-4.37 \tag{10}
\end{align*}
$$

This is a negative NPV project, and McGill Desautels should not start this new Master.
b. The value of the project after one good year, without reducing capacity, is

$$
\begin{align*}
P V & =e^{-0.1}(240 \times 0.54+150 \times 0.46)  \tag{11}\\
& =179.70 \tag{12}
\end{align*}
$$

The value of the project after one good year, if McGill reduces capacity, is

$$
\begin{align*}
P V & =e^{-0.1}(120 \times 0.54+75 \times 0.46)+75  \tag{13}\\
& =164.85 \tag{14}
\end{align*}
$$

Since $164.85<179.70$, McGill Desautels should not reduce capacity after one good year.
c. The value of the project after one bad year, without reducing capacity, is

$$
\begin{align*}
P V & =e^{-0.1}(150 \times 0.54+100 \times 0.46)  \tag{15}\\
& =114.91 \tag{16}
\end{align*}
$$

The value of the project after one bad year, if McGill reduces capacity, is

$$
\begin{align*}
P V & =e^{-0.1}(75 \times 0.54+50 \times 0.46)+75  \tag{17}\\
& =132.46 \tag{18}
\end{align*}
$$

Since $132.46>114.91$, McGill Desautels should reduce capacity after one bad year.
d. The new NPV of the project, with the option to reduce capacity is

$$
\begin{align*}
N P V & =e^{-0.1}(179.70 \times 0.54+132.46 \times 0.46)-140  \tag{19}\\
& =142.94-140  \tag{20}\\
& =2.94 \tag{21}
\end{align*}
$$

Since the new NPV is positive, McGill Desautels should start this new Master.

## 3 (40 points) Implied risk-neutral probabilities

This exercise shows how the prices of butterfly spreads can be used to determine the market's implied risk-neutral probabilities.

Consider buying a call option with an exercise price of $K_{1}=119$, selling two call options with an exercise price of $K_{2}=120$, and buying a call option with an exercise price of $K_{3}=121$. All these options mature in 4 years. The underlying stock is currently $\$ 99.9922$ and its volatility is $30 \%$ per annum. The continuously compounded risk-free rate is $4.56 \%$, and the dividend yield is zero.
a. (5 points) Plot the payoff for this "butterfly spread" strategy.

b. (5 points) Consider the triangle between the payoff of the butterfly spread and the horizontal axis on the plot. What is the area of this triangle?

> Area of the triangle
c. (10 points) You have decided to evaluate this strategy with a 4-period binomial tree. In table below, fill the payoffs of the strategy at the final five nodes of the tree, together with the corresponding final probabilities (I have filled in for your convenience the final values of the underlying, two payoffs, and two probabilities):

| State | $S_{4}$ | Payoff | Probability |
| :---: | :---: | :---: | :---: |
| $4 \times$ up, $0 \times$ down | $\$ 398.4140$ |  |  |
| $3 \times$ up, $1 \times$ down | $\$ 218.6542$ |  |  |
| $2 \times$ up, $2 \times$ down | $\$ 120.0000$ |  |  |
| $1 \times$ up, $3 \times$ down | $\$ 65.8574$ | $\$ 0$ | 0.3227 |
| $0 \times$ up, $4 \times$ down | $\$ 36.1433$ | $\$ 0$ | 0.1089 |

d. (10 points) What is the price of the butterfly spread today (at time 0 )?

> Price of the butterfly spread
e. (5 points) What is the future value of the butterfly spread at time 4? What do you observe? Discuss.

Future value of the butterfly spread
f. (5 points) What should be the future value at time 4 of a butterfly spread with strikes $K_{1}=64.8574, K_{2}=65.8574$, and $K_{3}=66.8574$ ? Why?

Future value of the butterfly spread

## 3 Solution

a. The payoff of the butterfly spread is

b. The area of the triangle is $\frac{1 \times 2}{2}=1$.
c. First, compute $u=1.4128, d=0.7754$, and $p^{*}=0.4256$. The complete table is

| State | $S_{4}$ | Payoff | Probability |
| :---: | :---: | :---: | :---: |
| $4 \times$ up, $0 \times$ down | $\$ 398.4140$ | $\$ 0$ | 0.0328 |
| $3 \times$ up, $1 \times$ down | $\$ 218.6542$ | $\$ 0$ | 0.1771 |
| $2 \times$ up, $2 \times$ down | $\$ 120.0000$ | $\$ 1$ | 0.3586 |
| $1 \times$ up, $3 \times$ down | $\$ 65.8574$ | $\$ 0$ | 0.3227 |
| $0 \times$ up, $4 \times$ down | $\$ 36.1433$ | $\$ 0$ | 0.1089 |

d. The price of the butterfly spread today is

$$
\begin{equation*}
\text { Price }=e^{-0.0456 \times 4} \times 1 \times 0.3586=0.2988 \tag{22}
\end{equation*}
$$

e. The future value of the butterfly spread is

$$
\begin{equation*}
\text { Future value }=1 \times 0.3586=0.3586 \tag{23}
\end{equation*}
$$

This is exactly the probability to reach the third final node (the one with $S_{4}=$ 120). Thus the market value of the butterfly spread can be used to determine implied risk-neutral probabilities.
f. According to the logic above, the future value of the butterfly spread should be the risk neutral probability of reaching the node $S_{4}=65.8574$, that is, 0.3227 .


[^0]:    ${ }^{1}$ In accord with McGill University's Charter of Students' Rights, students in this course have the right to submit in English or in French any written work that is to be graded. This does not apply to courses in which acquiring proficiency in a language is one of the objectives.

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    By signing the exam: ( $i$ ) you certify your presence, and (ii) you state that you neither gave nor received help during the examination.

