UCLA Anderson School of Management
Daniel Andrei, Derivative Markets 237D, Winter 2017

MFE – Final Exam

March 2017

Date: ______________________

Your Name: ______________________

Your email address: ______________________

Your Signature:[1] ______________________

- This exam is open book, open notes. You can use a calculator or a computer, **but be sure to show or explain your work**.
- You are not allowed to communicate with anyone (verbally, in writing, or electronically), except for me, during the exam period.
- You may present calculations in non-reduced form (e.g., as "$e^{0.095} - 1$").
- If you are stuck on something, make an assumption, tell me what it is, and do the best you can. I give partial credit if you provide enough correct information.

TIME LIMIT: 3 hours

TOTAL POINTS: 100

[1] As a member of the UCLA Anderson academic community, the highest standards of academic behavior are expected of you. It is your responsibility to make yourself aware of these standards (specifically regarding plagiarism, individual work, and team work) and adhere to them.
By signing the exam: (*i*) you certify your presence, and (*ii*) you state that you neither gave nor received help during the examination.

1 Various questions (32 points)

a. (4 points) A function is midpoint convex on an interval H if $\forall x_1, x_2 \in H$:

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}. \tag{1}$$

Furthermore, a continuous function that is midpoint convex is convex. Use this property to show that the price of a European option (call or put) is a convex function of the strike price.

b. (4 points) Consider two European put options, written on the same underlying and with the same maturity, with strikes \$50 and \$100. The price of the put struck at \$50 is \$1, whereas the price of the put struck at \$100 is \$2. Is there an arbitrage opportunity? If yes, clearly describe your arbitrage strategy.

c. (4 points) Suppose a stock price today is \$100. Which of these portfolios of European options is worth more?

(i) A long call struck at \$95, short two calls at \$100, and long a call at \$105

(ii) A long call stuck at \$95 and short a call struck at \$105

All options are written on the same underlying and have the same maturity.

d. (4 points) Theta represents time decay. As time goes on, your option is running out. That is usually a bad thing. Can you think of an example of an option that has positive theta?

Here is the forward curve for oil:

Maturity	Today=Spot	1 year	2 years	3 years	4 years
Price	44.00	56.00	59.00	62.00	70.00

The **effective annual** risk-free interest rates on zero-coupon bonds are: 1-year 1%, 2-years 2%, 3-years 3%, 4-years 4%.

e. (4 points) What is the annual implied forward interest rate $r_0(1,2)$? What is the annual implied forward interest rate $r_0(1,4)$?

$r_0(1,2)$:

$r_0(1,4)$:

f. (4 points) An industrial producer, IP Inc., needs to buy 1,000 barrels of oil in 1 year from today, 1,000 in 2 years, 1,000 in 3 years and 1,000 in 4 years from today. What is the 4-year swap price?

4-year swap price:

g. (4 points) Suppose that in order to hedge interest rate risk on your borrowing, you enter into an FRA that will guarantee a 3.01% effective annual interest rate for 1 year on \$1,000,000. On the date you borrow the \$1,000,000, the actual interest rate is 3.06%. Determine the dollar settlement of the FRA assuming settlement occurs on the date the loan is repaid.

Dollar settlement:

h. (4 points) Determine the dollar settlement of the above FRA assuming settlement occurs on the date the loan is initiated.

Dollar settlement:

1 *Answers:*

a. Consider a butterfly spread using calls, with strikes $K_1 < K_2 < K_3$, where $K_2 = \frac{K_1+K_2}{2}$. Because the payoff of such strategy is non-negative everywhere, its price today has to be positive. Thus, it must be that:

$$C(K_1) - 2C(K_2) + C(K_3) \geq 0, \tag{2}$$

which can also be written

$$C\left(\frac{K_1 + K_2}{2}\right) \leq \frac{C(K_1) + C(K_2)}{2}. \tag{3}$$

It follows from property (1) that the call is a convex function of the strike price.

b. Since the value of a put option with strike 0 is \$0, we in fact know the prices of of put options with three different strikes:

$$P(0) = 0; \quad P(50) = 1; \quad P(100) = 2. \tag{4}$$

The value of a butterfly spread formed with these three put options should be higher than zero, but given the prices above, the butterfly spread has a value of zero. Therefore, the put with strike 50 is relatively overpriced. This can be arbitraged by setting up the following portfolio:

- Long 1 put with strike 100
- Short 2 puts with strike 50

This portfolio has a value of zero today, but its payoff at maturity is positive or zero.

c. Denoting by $p(i)$ and $p(ii)$ the values of these portfolios, we have

$$p(i) - p(ii) = 2C(105) - 2C(100) < 0, \tag{5}$$

and thus the $p(ii)$ is more expensive.

d. A very deep in-the-money put can have a positive theta. As a extreme example, consider a put on an underlying asset that has gone to zero. This is as good as it gets for the holder of a put option. The value of the option is the present value of the strike and grows to reach the strike at maturity, i.e., the option has a positive theta.

e. Obtain first the prices of zero-coupon bonds:

Maturity t	0	1	2	3	4
P(0,t)		0.9901	0.9612	0.9151	0.8548

and therefore we obtain

$$\frac{1}{1 + r_0(1,2)} = \frac{P(0,2)}{P(0,1)} \tag{6}$$

$$r_0(1,2) = 3.01\% \tag{7}$$

and

$$\frac{1}{(1 + r_0(1,4))^3} = \frac{P(0,4)}{P(0,1)} \tag{8}$$

$$r_0(1,4) = 5.02\% \tag{9}$$

f. Use the formula

$$\bar{F} = \frac{\sum_{i=1}^{4} Q_{t_i} P(0, t_i) F_{0,t_i}}{\sum_{i=1}^{4} Q_{t_i} P(0, t_i)} \tag{10}$$

The quantities Q_{t_i} are the same. The 4-year swap price is \$61.47.

g. We have $r_{FRA} = 3.01\%$ and $r = 3.06\%$. The dollar settlement is

$$(r - r_{FRA}) \times 1,000,000 = \$500. \tag{11}$$

h. We have $r_{FRA} = 3.01\%$ and $r = 3.06\%$. The dollar settlement is

$$\frac{r - r_{FRA}}{1 + r} \times 1,000,000 = \$485.15. \tag{12}$$

2 Black-Scholes (8 points)

Consider a non-dividend-paying asset and consider a call and a put on the asset, both European and with the same strike and time to expiration. Denote by t today, T the expiry of the option, and $r > 0$ the continuously compounded risk-free rate. The price of the asset today is S_t. Suppose we use Black-Scholes to price options.

a. (4 points) If the common strike on the two options is set to $S_t e^{r(T-t)}$, which of the two options is worth more? Justify your answer.

b. (4 points) If the common strike on the two options is set to S_t, which of the two options is worth more? Justify your answer.

2 *Answers:*

a. From the put-call parity, we know that the price of a call equals the price of a put with the same maturity when the strike price is equal to the forward price. In this case, the forward price is

$$F_{t,T} = S_t e^{r(T-t)}, \tag{13}$$

because the asset is not paying dividends. Thus, the two options have the same price.

b. Let C^a and P^a the call and the put option from the previous point. We know that $C^a = P^a$. Consider now the options C^b and P^b, both with strike S_t, which is lower than $S_t e^{r(T-t)}$ as long as $r > 0$. Since the strike is lower, we have:

$$C^b > C^a \tag{14}$$

$$P^b < P^a \tag{15}$$

and thus $C^b > P^b$. The call option is worth more.

3 Vega (12 points)

Vega is the option's sensitivity to a small change in the volatility of the underlying asset. Vega is identical for put and call options and is greater than zero:

$$\text{Vega}_{\text{call, put}} = S_0 e^{-\delta T} N'(d_1)\sqrt{T},$$

where $N'(d_1) = e^{-d_1^2/2}/\sqrt{2\pi}$.

When you are trying to profit from moves in implied volatility, it is useful to know where the option attains its *maximum vega*.

a. (3 points) What is the value of vega as the stock price approaches zero? Justify your answer.

$\lim_{S_0 \to 0}$ Vega:

b. (3 points) What is the value of vega as the stock price approaches infinity? You can justify your answer with calculus, but an intuitive answer is enough here.

$\lim_{S_0 \to \infty}$ Vega:

c. (3 points) What is the asset price that maximizes vega given the strike price X? Justify your answer.

Asset price:

d. (3 points) What is the strike price that maximizes vega given the asset price S_0? Justify your answer.

Strike price:

3 *Answers:*

a. It is easy to see that the limit is zero.

b. The limit should be zero as well. Consider a call option: if the stock approaches infinity, then the option is deep in the money and changes in volatility do not matter for the final payoff.

c. Replace d_1 in

$$\text{Vega}_{\text{call, put}} = S_0 e^{-\delta T} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \sqrt{T}, \tag{16}$$

then take the derivative with respect to S_0. This will yield:

$$S_0 = X e^{\left(\delta - r + \frac{\sigma^2}{2}\right)T}. \tag{17}$$

d. Similarly,

$$X = S_0 e^{\left(r-\delta+\frac{\sigma^2}{2}\right)T}. \tag{18}$$

4 Butterfly Spread and Calendar Spread (20 points)

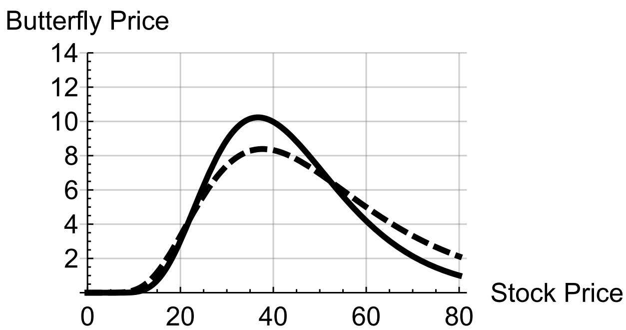
The price of a non-dividend paying stock today is \$40. Suppose the interest rate is 9% (continuously compounded) and the stock return volatility is 28%. Consider the following European call options:

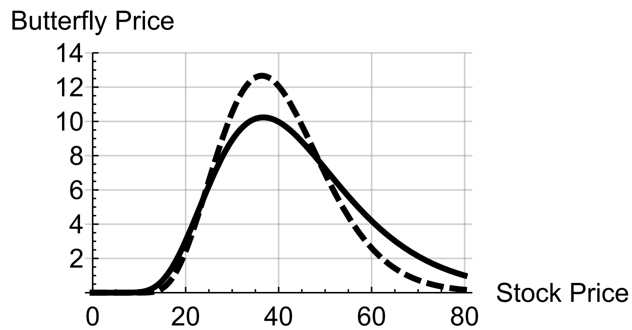
	Call #1	Call #2	Call #3	Call #4
Maturity (years)	1	1	1	0.5
Strike	20	40	60	40
Call Price	21.73	6.20	0.85	4.04
Delta	0.9983	0.6778	0.1619	0.6279
Vega	0.0021	0.1435	0.0981	0.1070
Theta	-0.0046	-0.0107	-0.0051	-0.0134

a. (4 points) Suppose you build a butterfly spread with the calls #1, #2, and #3. How many units of the stock you need to buy/sell in order to delta-hedge the butterfly spread?

Units of the stock:

b. (4 points) The solid lines in the two plots below show the price of the butterfly spread as a function of the stock price. Suppose that volatility goes up to 40%. In the plots below, which one of the two dashed lines corresponds to the correct movement of the butterfly spread price? Justify your answer.





c. (4 points) Suppose you build a calendar spread with the calls #2 and #4. How many units of the stock you need to buy/sell in order to delta-hedge the calendar spread?

Units of the stock:

d. (4 points) The solid lines in the two plots below show the price of the calendar spread as a function of the stock price. Suppose that volatility goes up to 40%. In the plots below, which one of the two dashed lines corresponds to the correct movement of the calendar spread price? Justify your answer.

e. (4 points) The solid lines in the two plots below show the price of the calendar spread as a function of the stock price. Suppose that three months have passed by but nothing else has changed. In the plots below, which one of the two dashed lines corresponds to the correct movement of the calendar spread price? Justify your answer.

4 *Answers:*

a. The delta of the butterfly spread is

$$0.9983 - 2 \times 0.6778 + 0.1619 = -0.1953. \tag{19}$$

Thus, we need to buy 0.1953 units of the stock in order to delta-hedge the butterfly spread.

b. The vega of the butterfly spread is

$$0.0021 - 2 \times 0.1435 + 0.0981 = -0.1867 \tag{20}$$

Because the vega is negative, an increase in volatility decreases the price of the calendar spread. Thus, the correct movement of the butterfly spread price is the one on the left hand side.

c. The delta of the calendar spread is

$$0.6778 - 0.6279 = 0.0499 \tag{21}$$

Thus, we need to sell 0.0499 units of the stock in order to delta-hedge the calendar spread.

d. The vega of the calendar spread is

$$0.1435 - 0.1070 = 0.0365 \tag{22}$$

Because the vega is positive, an increase in volatility increases the price of the calendar spread. Thus, the correct movement of the calendar spread price is the one on the left hand side.

e. The theta of the calendar spread is

$$-0.0107 + 0.0134 = 0.0027 \tag{23}$$

Because the theta is positive, approaching maturity increases the price of the calendar spread. Thus, the correct movement of the calendar spread price is the one on the left hand side.

5 Option to Abandon (12 points)

Assume that Blue Star Aircraft is interested in building a small passenger plane and that it approaches Boeing with a proposal for a joint venture. Each firm will invest \$500 million in the joint venture. The investment is expected to have a 30-year lifespan.

Boeing works through a traditional investment analysis and concludes that their share of the present value of expected cash flows would be only \$480 million. The net present value of the project would therefore be negative (NPV=-\$20 million) and Boeing rejects this joint venture.

On rejection of the joint venture, Blue Star approaches Boeing with a sweetener, offering the option to buy out Boeing's 50% share of the joint venture after 5 years from today for \$400 million. Although this is less than what Boeing will invest initially, it puts a floor on the losses and thus gives Boeing an *abandonment option.*

a. (4 points) Value the abandonment option. Use a standard deviation of $\sigma = 0.25$ and a dividend yield of $\delta = 0.03$. The riskless rate is 5% (continuously compounded).

Abandonment option:

b. (4 points) Given this additional option, should Boeing enter this joint venture? Why?

c. (4 points) Assume now that Blue Star offers to buy out Boeing's 50% share of the joint venture *at any time over the next 5 years* for $400 million. Given this new option, should Boeing enter the joint venture? Why?

5 *Answers:*

a. The abandonment option is a European put option with strike $400 million. It's value is

$$P(480, 400, 0.25, 5\%, 5, 0.03) = \$38.5806 \text{ million.} \tag{24}$$

b. Boeing should enter into this joint venture, because the value is greater than the negative net present value of the investment (NPV=-$20 million). The fact that the option to abandon has value provides a rationale for Boeing to build the operating flexibility to terminate the project if it does not measure up to expectations.

c. The abandonment option is now an American put option with the same strike. Since an American option is always greater or equal than the European option with the same characteristics, this deal is now even better than before. Boeing should definitely enter into this joint venture.

6 Chooser Option (16 points)

A *chooser option* is a European option that matures at time T, but the buyer does not have to determine whether the option is a put or a call at the time of purchase. The buyer has until time t to make his choice, where $t < T$. Assume that the call and the put both have the same strike price and time to expiry, and we will assume that the underlying asset pays no dividends.

The value of a chooser option at any time after the choice has been made (i.e., between times t and T) is just the value of a European call or put, depending on the choice made by the holder.

Define $c(X, t, T)$ as the value of a European call option at time t with a strike of X that expires at time T. Notice that the time remaining until maturity is $T - t$. Define $p(X, t, T)$ analogously.

At time t, the owner of the chooser will rationally choose whichever of the call or put has greater value. Thus, at time t, the value of the chooser is

$$\max\left[c(X, t, T), p(X, t, T)\right].$$

a. (4 points) Show that the value of the chooser today (at time 0) is given by

$$c(X, 0, T) + p(Xe^{-r(T-t)}, 0, t). \tag{25}$$

b. (4 points) Show that the value of the chooser today (at time 0) is given by

$$p(X, 0, T) + c(Xe^{-r(T-t)}, 0, t). \tag{26}$$

c. (4 points) We have now derived two expressions for the value today of the chooser option, one as in Equation (25) and the other as in Equation (26). Show that these two formulations are equivalent *using put-call parity arguments*.

d. (4 points) A *straddle* is a position that is long both a call and a put with the same strike and the same expiry. The value of the straddle that matures at time T is

$$c(X, 0, T) + p(X, 0, T).$$

Show that the value of the chooser option as given by either Equation (25) or (26) is less than the value of the straddle.

6 *Answers:*

a. Use the put-call parity to write

$$\begin{aligned} \max\left[c(X,t,T), p(X,t,T)\right] &= \max\left[c(X,t,T), c(X,t,T) + (Xe^{-r(T-t)} - S_t)\right] \\ &= c(X,t,T) + \underbrace{\max\left[0, Xe^{-r(T-t)} - S_t\right]}_{\substack{\text{Put option with strike} \\ Xe^{-r(T-t)}}} \end{aligned} \tag{27}$$

and thus the value of the chooser today (at time 0) is given by

$$c(X, 0, T) + p(Xe^{-r(T-t)}, 0, t). \tag{28}$$

b. Use an analogous argument as in point a.

c. Take the difference between (25) and (26):

$$\underbrace{c(X,0,T)-p(X,0,T)}_{S_0-Xe^{-rT}}-\underbrace{\left[c(Xe^{-r(T-t)},0,t)-p(Xe^{-r(T-t)},0,t)\right]}_{\substack{S_0-Xe^{-r(T-t)}e^{-rt}\\ =S_0-Xe^{-rT}}} \tag{29}$$

and one can clearly see that the difference is zero.

d. Compute

$$\text{Straddle} - \text{Chooser} = p(X,0,T) - p(Xe^{-r(T-t)},0,t) \tag{30}$$

The two put options on the right hand side are equal only in the extreme case when the stock price is zero (and their value is $Xe^{-r(T-t)}$). Otherwise, the first put option is more expensive because it has a higher strike. Thus, the straddle is more expensive than the chooser.