McGill Desautels Faculty of Management

Final Exam

December 2018

## Date:

$\qquad$

## Your Name:

$\qquad$

## Your Signature: ${ }^{1}$

- This exam is open book, open notes. You can use a calculator or a computer, but be sure to show or explain your work.
- You are not allowed to communicate with anyone (verbally, in writing, or electronically), except for me, during the exam period.
- You may present calculations in non-reduced form (e.g., as " $e^{0.095}-1$ ").
- If you are stuck on something, make an assumption, tell me what it is, and do the best you can. I give partial credit if you provide enough correct information.


## TIME LIMIT: 3 hours

TOTAL POINTS: 100

[^0]
## 1 (27 points) Warm-up questions

a. (3 points) The Black-Scholes formula for put options on non-dividend paying assets is given by

$$
\begin{equation*}
P_{0}=K e^{-r T} N\left(-d_{2}\right)+S_{0}\left(-N\left(-d_{1}\right)\right) . \tag{1}
\end{equation*}
$$

What do the values $N\left(-d_{2}\right)$ and $-N\left(-d_{1}\right)$ represent? Provide an economic interpretation.
b. (3 points) If the volatility decreases, all else equal, how do the call and put option prices change, respectively?
c. (3 points) For European put options on non-dividend paying assets, what are the smallest and largest values that Delta can take?
d. (3 points) When are a European call and an European put worth the same? (The options are written on the same asset and have the same strike and maturity.)
e. (3 points) What is the two year volatility of an asset with $30 \%$ three months volatility?
f. (3 points) Calculate the delta of an at-the-money six-month European call option on a non-dividend-paying stock when the risk-free interest rate is $10 \%$ per annum and the stock price volatility is $25 \%$ per annum.
g. (3 points) Explain briefly why portfolio insurance may have played a part in the Black Monday crash of October 19, 1987.
h. ${ }^{* * *} 3$ points) The time value of a call option is the difference between the value $C_{t}$ of the option and the intrinsic value $\max \left(S_{t}-K, 0\right)$ of the call option. In other words, the time value of the option is

$$
\begin{equation*}
C_{t}-\max \left(S_{t}-K, 0\right) \tag{2}
\end{equation*}
$$

Show that the time value of a European call option is highest at the money.
i. ( ${ }^{* * *} 3$ points) Show that the Black-Scholes formula for a European call option gives a price that tends to $\max \left(S_{0}-K, 0\right)$ as $T \rightarrow 0$.

## 1 Solution

a. $N\left(-d_{2}\right)$ is the risk-neutral probability that the put option will be exercised at maturity. $-N\left(-d_{1}\right)$ is the Delta of the put option (the sensitivity of the put option price with respect to fluctuations in the price of the underlying).
b. Both the call and the put prices decrease (the vega of both options is positive).
c. For call options $\Delta \in[0,1]$. For put options, $\Delta \in[-1,0]$.
d. Start from the put call parity:

$$
\begin{equation*}
C_{0}-P_{0}=S_{0} e^{-\delta T}-K e^{-r T} \tag{3}
\end{equation*}
$$

For the left hand side to be zero, we need $K=S_{0} e^{(r-\delta) T}$, or the forward price.
e. Use the square root of time rule (there are 8 three-month periods in two years):

$$
\begin{equation*}
0.3 \times \sqrt{8}=84.85 \% \tag{4}
\end{equation*}
$$

f. In this case $\Delta=N\left(d_{1}\right)$, with

$$
\begin{equation*}
d_{1}=\frac{\left(r+\sigma^{2} / 2\right)}{\sigma} \sqrt{T}=0.3712 \tag{5}
\end{equation*}
$$

and thus $N\left(d_{1}\right)=0.6448$.
g. Portfolio insurance can generate fire sales if implemented by a majority of market players. When the price of the underlying drops, the strategy requires further selling the underlying, which can put downward pressure on prices, which leads to further selling, and so on. This is perhaps one of the mechanisms that took place during October 19, 1987.
h. Consider two cases:
(i) The option is out of the money $\left(S_{t}<K\right)$. In this case the time value is $C_{t}$ and increases in $S_{t}$.
(ii) The option is in the money, in which case the time value is

$$
\begin{equation*}
C_{t}-S_{t}+K \tag{6}
\end{equation*}
$$

Since the delta of the call is lower than one, the time value now decreases in $S_{t}$. It follows that the time value of a European call is highest at the money.
i. There are two cases:
(i) When $S_{0}<K$, both $d_{1}$ and $d_{2}$ tend to $-\infty$. That makes $N\left(d_{1}\right)=N\left(d_{2}\right)=$ 0 and thus the value of the option is zero.
(ii) When $S_{0}>K$, both $d_{1}$ and $d_{2}$ tend to $\infty$. That makes $N\left(d_{1}\right)=N\left(d_{2}\right)=1$ and thus the value of the option is $S_{0}-K$.

We therefore obtain $\max \left(S_{0}-K, 0\right)$.

## 2 (9 points) Corporate acquisition

Firm $A$ has a stock price of $\$ 40$, and has made an offer for firm $B$ where $A$ promises to pay 1.5 shares for each share of $B$, as long as $A$ 's stock price remains between $\$ 35$ and $\$ 45$. If the price of $A$ is below $\$ 35$, $A$ will pay $\$ 52.50$ per share, and if the price of $A$ is above $\$ 45, A$ will pay $\$ 67.50$ per share. The deal is expected to close in 9 months. Assume (for $A$ 's stock): $\sigma=35 \%, r=5 \%$, and $\delta=0$.
a. (3 points) How are the values $\$ 52.50$ and $\$ 67.50$ arrived at?
b. (3 points) In the plot below, draw the value of the $A$ offer for one share of $B$, as a function of $A$ 's share price.

c. ( ${ }^{* * *} 3$ points) What is the value of the offer?

## 2 Solution

a. $52.5=35 \times 1.5$ and $67.5=45 \times 1.5$.
b. The graph below shows the value of the offer as a function of $A$ 's stock price.

c. The portfolio that replicates the payoff above is:
(i) A risk-free payment with face value 52.5
(ii) Buy 1.5 call options with strike 35
(iii) Sell 1.5 call options with strike 45
with value

$$
\begin{equation*}
50.5677+1.5 \times 8.2651-1.5 \times 3.5216=\$ 57.68 \tag{7}
\end{equation*}
$$

The value of the offer is thus $\$ 57.68$.

## 3 (12 points) Option portfolio Greeks

You are a risk analyst at St. Laurent LLP, a hedge fund in Ville-Marie. When the market closes, you start to examine the risk of the portfolio of Mr. Ponzi, a fund manager in your company. You log into the trading system and find the information shown in the table below. The trading system displays the individual portfolio positions and their corresponding Greeks (options are written on 1 unit of the underlying). The current price of the underlying asset is $\$ 31$. Assume $r=5 \%$.

|  | Put \#1 | Put \#2 | Put \#3 |
| :--- | ---: | ---: | ---: |
| Nb. of contracts | 100 | -200 | 100 |
| Years to maturity | 1 | 1 | 1 |
| Strike | 20 | 30 | 40 |
| Put Price | 0.1587 | 2.4571 | 8.5604 |
| Delta | -0.0380 | -0.3351 | -0.7025 |
| Gamma | 0.0089 | 0.0391 | 0.0372 |
| Vega | 0.0256 | 0.1130 | 0.1074 |
| Theta | -0.0009 | -0.0029 | -0.0003 |
| Rho | -0.0134 | -0.1285 | -0.3034 |

a. (4 points) Which of the positions in the portfolio is the most sensitive to changes in the volatility of the underlying asset? Why?
b. ( ${ }^{* * *} 4$ points) Suppose that the portfolio is held over the next month. If during this time period the underlying asset price decreases by $\$ 1$ and the risk-free rate decreases by $1 \%$ (one percentage point), what is the approximate gain or loss of the portfolio over the next month (i.e., after 30 days)?
c. (4 points) Mr. Ponzi's portfolio corresponds to a common option strategy. In the plot below, draw the portfolio's profit diagram (using prices in the table to calculate the future value of the premium for the portfolio). What is the height of the plot at $S_{T}=\$ 30$ ? Clearly indicate this point on the diagram. Comment on the risk characteristics of the strategy (describe when it benefits and when it loses).


## 3 Solution

a. The vega is largest for the Put $\# 2$, which is sold short in 200 units. This is the position most sensitive to changes in the volatility of the underlying asset.
b. We have the following changes:
(i) $S$ decreases by $1 \$$
(ii) $r$ decreases by $1 \%$
(iii) 30 days have passed by

We need therefore three greeks: Delta, Rho, and Theta. For this portfolio, these Greeks are:

$$
\begin{align*}
\text { Delta } & =-7.0130  \tag{8}\\
\text { Rho } & =-5.9790  \tag{9}\\
\text { Theta } & =0.4645 \tag{10}
\end{align*}
$$

The approximate change in the portfolio is therefore

$$
\begin{equation*}
(-7.0130) \times(-1)+(-5.9790) \times(-1)+0.4645 \times 30=26.93 \tag{11}
\end{equation*}
$$

Thus, there is a gain of $\$ 26.93$.
c. The price of this portfolio is $\$ 380.492$. The future value of the premium is thus $380.492 \times e^{0.05}=400$. This is a butterfly spread, whose profit diagram is provided below. The height at 30 is 600 .


This strategy benefits when the volatility decreases and the underlying remains close to 30 . It loses when volatility increases and/or the underlying price makes large movements far from 30.

## 4 (9 points) Digital put

The company $i$ FakeNews is specialized in creating products and services that are on the forefront of the next generation of transformative technologies worldwide. Its shares are listed permanently and are worth currently $\$ 40$ on the market. The annual volatility of their return is $5 \%$. The risk-free interest rate is $5 \%$ per annum. iFakeNews cannot afford to pay any dividends $(\delta=0)$.

A digital put option is an option that pays its holder an amount of $\$ 1$ if the value of the underlying at maturity is less than the strike price, and $\$ 0$ otherwise. The value in period 0 of a digital put option maturing in period $T$ with a strike price of $K$ equals

$$
\begin{equation*}
P_{0}^{d}=e^{-r T} N\left(-d_{2}\right), \tag{12}
\end{equation*}
$$

where $N(\cdot)$ represents the cumulative standard normal distribution and $d_{2}$ is defined as in the Black-Scholes formula,

$$
\begin{equation*}
d_{2}=\frac{\ln \left(S_{0} / K\right)+\left(r-\sigma^{2} / 2\right) T}{\sigma \sqrt{T}} . \tag{13}
\end{equation*}
$$

a. (3 points) Draw the payoff diagram of a digital put option on the shares of $i$ FakeNews with an exercise price of 45 .

b. (3 points) What is the value of a digital put option on $i$ FakeNews maturing in 2 years and having an exercise price of 45 ?
c. ( ${ }^{* * *} 3$ points) You hold 45 digital put options on $i$ FakeNews maturing in 2 years and having an exercise price of 45 . What is the delta of your position? Hint: $\partial N(z) / \partial z$ is the probability density function of the standard normal distribution:

$$
\begin{equation*}
\frac{\partial N(z)}{\partial z}=N^{\prime}(z)=\frac{1}{\sqrt{2 \pi}} e^{-z^{2} / 2} \tag{14}
\end{equation*}
$$

The Excel formula for $N^{\prime}(z)$ is NORM.S.DIST $(z, 0)$.

## 4 Solution

a. The payoff diagram is:

b. With the above parameters, obtain $d_{2}=-0.2869$. Thus

$$
\begin{equation*}
P_{0}^{d}=e^{-0.05 \times 2} N(0.2869)=\$ 0.5546 \tag{15}
\end{equation*}
$$

c. The delta of an individual digital put is found by differentiation

$$
\begin{equation*}
\Delta=\frac{\partial P_{0}^{d}}{\partial S_{0}}=-\frac{1}{S_{0} \sigma \sqrt{T}} e^{-r T} N^{\prime}(-d 2)=-0.12248 \tag{16}
\end{equation*}
$$

The delta of the position is therefore $45 \times(-0.12248)=-5.5116$.

## 5 (15 points) Straddle and replication

You have just started your new internship at $i$ Risk, in Montréal, on June 1st 2018. This date is considered to be $t=0$. On the same day, the CEO (Marissa) wants to see you in order to ask for advice on a new investment strategy.

Marissa tells you that the S\&P 500 market index has reached a level of $\$ 2,500$ and that there will be a lot of volatility in the markets for the next two years, but without being able to guess a downward or upward trend. Very enthusiastic, you suggest that in these circumstances a "straddle" strategy works best, with maturity $T=2$ and exercise price $K=\$ 2,500$.

Marissa completely trusts your knowledge, but she asks for additional details.
a. (3 points) Using the plot below, draw the payoff of this strategy at maturity.

b. (3 points) You tell Marissa that two more parameters are needed to take the analysis further: the risk-free rate $r$ (in annual terms, continuously compounded) and the volatility of the returns of the S\&P 500 index (in annual terms). She happily provides the risk-free rate $r=0.05$, but she tells you to compute the volatility yourself.

Describe what you would do to obtain the volatility. What data will you download? What formula would you use?
c. (3 points) After a few hours of work, you find $\sigma=0.25$. You also know the dividend yield of the $\mathrm{S} \& \mathrm{P} 500, \delta=3 \%$. What is the price of your strategy today?
d. (3 points) It's now $6: 30 \mathrm{pm}$ and you decide to go home, after a great first day. Sadly, Marissa finds out that there are no options available on the market with this exercise price and this maturity. Marissa is disappointed. She asks you to stay at work all night and take short naps in your cubicle, under the desk. You have a better solution: you will replicate the strategy.

What is the replicating portfolio of the strategy? Describe the positions in the S\&P 500 and in the risk-free asset.
e. (3 points) Three months later, Marissa wonders whether your dynamic portfolio needs to be adjusted. She thinks that this should not be the case, since the S\&P 500 happens to be at the same level, and the volatility is still 0.25 . Is Marissa right? Justify your answer without doing any calculus.

## 5 Solution

a. The payoff diagram is on the next page.

b. There are two possible answers here. You can use historical data and directly compute the standard deviation of returns:

$$
\begin{equation*}
\sigma=\sqrt{\frac{1}{N} \sum_{t=0}^{T}\left(r_{t}-\mu\right)^{2}} \tag{17}
\end{equation*}
$$

Alternatively, you can use other options on the same underlying to compute the implied volatility.
c. The straddle is a call plus a put. The call price is $\$ 372.09$. The put price is $\$ 279.78$. Thus, the price of the strategy today is $\$ 651.87$.
d. The delta of the straddle is the delta of the call plus the delta of the put:

$$
\begin{equation*}
\Delta=0.5783-0.3635=0.2148 \tag{18}
\end{equation*}
$$

Thus, you will buy 0.2148 units of S\&P 500, which makes a total of $\$ 537.08$. The rest (\$114.78) you invest in the risk-free asset.
e. Marissa is wrong, because the delta of the asset has certainly changed. You will have to adjust your dynamic portfolio.

## 6 (12 points) Abandonment option

Assume that Blue Star Aircraft is interested in building a small passenger plane and that it approaches Boeing with a proposal for a joint venture. Each firm will invest $\$ 500$ million in the joint venture. The investment is expected to have an infinite lifespan.

Boeing works through a traditional investment analysis and concludes that their share of the present value of expected cash flows would be only $\$ 470$ million. The net present value of the project would therefore be negative (NPV $=-\$ 30$ million) and Boeing rejects this joint venture.

On rejection of the joint venture, Blue Star approaches Boeing with a sweetener, offering the option to buy out Boeing's share of the joint venture at any time into the future starting from today, for $\$ 300$ million. Although this is much less than what Boeing will invest initially, it puts a floor on the losses and thus gives Boeing an abandonment option (that is, a perpetual put option).
a. (4 points) Value the abandonment option. Use a standard deviation of $\sigma=$ 0.3 and a dividend yield of $\delta=1 / 30$. The risk-free rate is $5 \%$ (continuously compounded).
b. (4 points) Given this additional option, should Boeing enter this joint venture? Justify your answer.
c. (4 points) Assume that Boeing enters this joint venture. According to calculations made at point $a$, when should Boeing "abandon" the project and sell back its share?

## 6 Solution

a. The abandonment option is a perpetual put with strike $\$ 300$ million, with value

$$
\begin{equation*}
\text { PUTPERPETUAL }(470,300,0.3,5 \%, 1 / 30)=\$ 61.97 \text { million. } \tag{19}
\end{equation*}
$$

b. Boeing should enter into this joint venture, because the value is greater than the negative net present value of the investment (NPV $=-\$ 30$ million). The fact that the option to abandon has value provides a rationale for Boeing to build the operating flexibility to terminate the project if it does not measure up to expectations.
c. The barrier of the perpetual put is $\$ 131.96$ million. Boeing should abandon the project when the present value of expected cash flows touches this barrier.

7 (16 points) Bitcoin and blockchain. Choose the (unique) correct answer:
a. (2 points) Which of the following hexadecimal numbers is larger?

X=B43373E865748CEE3D06CAC82D2C3EAAD2648C634B05CEAA848F09547E24EC6D
Y=B43373E855748CEE3D06CAC82D2C3EAAD2648C634B05CEAA848F09547E24EC6D
(i) X
(ii) Y
(iii) It's impossible to tell
b. (2 points) What incentivizes the miners to give correct validation of transactions?
(i) A block reward
(ii) A nonce
(iii) Thumbs up from the community
(iv) More memory
c. (2 points) What is a hash function?
(i) A function that takes an input of any length and returns a fixed-length string of numbers and letters
(ii) A culinary dish
(iii) A fork
(iv) Gas
d. (2 points) How many Bitcoin will ever be created?
(i) $21,000,000$
(ii) $3,500,000$
(iii) Unlimited
(iv) $18,650,000$
e. (2 points) Which traditional stock exchange was the first to list Bitcoin futures contracts?
(i) The Chicago Board Options Exchange (CBOE)
(ii) The New York Stock Exchange (NYSE)
(iii) The Intercontinental Exchange (ICE)
(iv) None of the above. Futures contracts are only available on cryptocurrency exchanges like BitMex and Bitfinex.
f. (2 points) Where is the Bitcoin processing server located?
(i) Bitcoin has no processing server
(ii) Washington, D.C., USA
(iii) London, England
(iv) In Satoshi Nakamoto's basement
g. (2 points) How often, on average, can we expect a new block to be found by miners?
(i) 10 minutes
(ii) 1 second
(iii) 2 minutes
(iv) 60 minutes
h. (2 points) What is "difficulty" in relation to Bitcoin?
(i) A measure of how difficult it is to find a hash below the target
(ii) A measure of how hard it is to explain what Bitcoin is
(iii) A measure of how hard it is for Bitcoin to recover to its all-time high
(iv) A measure of long it takes to send Bitcoin between addresses

## 7 Solution

The correct answer is (i) for all questions.


[^0]:    ${ }^{1}$ In accord with McGill University's Charter of Students' Rights, students in this course have the right to submit in English or in French any written work that is to be graded. This does not apply to courses in which acquiring proficiency in a language is one of the objectives.

    McGill University values academic integrity. Therefore, all students must understand the meaning and consequences of cheating, plagiarism and other academic offenses under the Code of Student Conduct and Disciplinary Procedures: https://www.mcgill.ca/students/srr/honest.

    By signing the exam: (i) you certify your presence, and (ii) you state that you neither gave nor received help during the examination.

