Investments - Midterm

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Solution

Duration: 2 hours. Documentation and calculators authorized

Problem 1 (Multiple Choice, 30 points)

One correct answer per question. You do not need to justify your answer.

1. An analyst estimates that a stock has the following probabilities of return depending on the state of the economy:

State of Economy	Probability	Return
Good	0.1	15%
Normal	0.6	13%
Poor	0.3	7%

The expected return of the stock is:

(a) 7.8%
(b) **11.4%**
(c) 11.7%
(d) 13.0%

2. The variable a in the utility formula $U = \mu - \frac{a}{2}\sigma^2$ represents the:

(a) investor's return requirement
(b) **investor's aversion to risk**
(c) certainty equivalent rate of the portfolio
(d) preference for one unit of return per four units of risk.

3. The change from a straight to a kinked capital allocation line is a result of the:

(a) reward-to-variability ratio increasing
(b) **borrowing rate exceeding the lending rate**
(c) investor's risk tolerance decreasing
(d) increase in the portfolio proportion of the risk-free asset

4. You manage an equity fund with an expected risk premium of 10% and an expected standard deviation of 14%. The rate on Treasury bills is 6%. Your client chooses to invest \$60'000 of her portfolio in your equity fund and \$40'000 in a T-bill money market fund. What is the expected return and standard deviation of return on your client's portfolio?

   (a) 8.4% and 8.4%
   (b) 8.4% and 14%
   (c) **12% and 8.4%**
   (d) 12% and 14%

5. What is the reward-to-variability ratio for the *equity fund* in the previous problem?

   (a) **0.71**
   (b) 1.00
   (c) 1.19
   (d) 1.91

6. Which statement about portfolio diversification is correct?

   (a) proper diversification can reduce or eliminate systematic risk
   (b) diversification reduces the portfolio's expected return because it reduces a portfolio's total risk
   (c) **as more securities are added to a portfolio, total risk typically would be expected to fall at a decreasing rate**
   (d) the risk-reducing benefits of diversification do not occur meaningfully until at least 30 individual securities are included in the portfolio.

7. Portfolio theory as described by Markowitz is most concerned with:

   (a) the elimination of systematic risk
   (b) **the effect of diversification on portfolio risk**
   (c) the identification of unsystematic risk
   (d) active portfolio management to enhance return

8. The security market line depicts:

   (a) **a security's expected return as a function of its systematic risk**
   (b) the market portfolio as the optimal portfolio of risky securities
   (c) the relationship between a security's return and the return on an index
   (d) the complete portfolio as a combination of the market portfolio and the risk-free asset.

9. Capital asset pricing theory asserts that portfolio returns are best explained by:
   (a) economic factors
   (b) specific risk
   (c) **systematic risk**
   (d) diversification

10. According to CAPM, the expected rate of return of a portfolio with a beta of 1.0 and an alpha of 0 is:
   (a) between r_M and r_f
   (b) the risk-free rate r_f
   (c) $\beta (r_M - r_f)$
   (d) **the expected return on the market,** r_M

11. According to the theory of arbitrage:
   (a) high-beta stocks are consistently overpriced
   (b) low-beta stocks are consistently overpriced
   (c) **positive alpha investment opportunities will quickly disappear**
   (d) rational investors will pursue arbitrage consistent with their risk tolerance

12. The arbitrage pricing theory (APT) differs from the single-factor capital asset pricing model (CAPM) because the APT:
   (a) places more emphasis on market risk
   (b) minimizes the importance of diversification
   (c) recognizes multiple unsystematic risk factors
   (d) **recognizes multiple systematic risk factors**

13. In contrast to CAPM, APT:
   (a) requires that markets be in equilibrium
   (b) uses risk premiums based on micro variables
   (c) specifies the number and identifies specific factors that determine expected returns
   (d) **does not require the restrictive assumption concerning the market portfolio**

14. If the exchange rate value of the British pound goes from U.S.$1.80 to U.S.$1.60, then the pound has:
   (a) appreciated and the British will find U.S. goods cheaper
   (b) appreciated and the British will find U.S. goods more expensive

(c) **depreciated and the British will find U.S. goods more expensive**

(d) depreciated and the British will find U.S. goods cheaper

15. The Gordon model will not produce a finite value if the dividend growth rate is:

    (a) above its historical average

    (b) **above the market capitalization rate**

    (c) below its historical average

    (d) below the market capitalization rate

16. A stock has a required return of 15%, a constant-growth rate of 10%, and a dividend payout ratio of 45%. The stock's price-earnings ratio should be:

    (a) 3

    (b) 4.5

    (c) **9**

    (d) 11

Problem 2 (Two-Fund Separation, 30 points)

You know that the composition of the minimum variance portfolio with expected return μ_p is given by

$$w = \lambda \Sigma^{-1}\mathbf{1} + \gamma \Sigma^{-1}\mu, \tag{1}$$

with the usual notation from the class notes. The global minimum variance portfolio has $\lambda_g = \frac{1}{A}$, and $\gamma_g = 0$, thus

$$w_g = \lambda_g \Sigma^{-1}\mathbf{1} = \frac{\Sigma^{-1}\mathbf{1}}{A}. \tag{2}$$

Its expected return is $\mu_g = \frac{B}{A}$, and its variance of returns is $\sigma_g^2 = \frac{1}{A}$.

1. Find the composition of a minimum variance portfolio with expected return $\mu_d = \frac{C}{B}$. Proceed by computing λ_d and γ_d first, then compute w_d.
   Solution
   The solutions for λ_d and γ_d are

   $$\lambda_d = 0, \quad \gamma_d = \frac{1}{B} \tag{3}$$

   It follows that the optimal portfolio is

   $$w_d = \lambda_d \Sigma^{-1}\mathbf{1} + \gamma_d \Sigma^{-1}\mu = \frac{\Sigma^{-1}\mu}{B} \tag{4}$$

2. Verify that $\lambda A+\gamma B=1$. Try to write the composition w of any minimum variance portfolio as a function of w_g and w_d. What do you conclude? Is this result useful? Why?
Solution

$$\lambda A+\gamma B=\frac{(C-\mu_p B)\,A+(\mu_p A-B)\,B}{\Delta}=\frac{CA-B^2}{\Delta}=1 \tag{5}$$

Thus, the composition of any minimum variance portfolio is

$$w=\lambda A\frac{\Sigma^{-1}\mathbf{1}}{A}+\gamma B\frac{\Sigma^{-1}\mu}{B}=\lambda A w_g+\gamma B w_d \tag{6}$$

Interpretation: All minimum variance portfolios can be obained as a combination of the global minimum-variance portfolio w_g and portfolio w_d (the two-fund separation result).

3. Compute the variance of returns for the portfolio w_d.
Solution

$$\sigma_d^2=w_d'\Sigma w_d=\frac{\mu'\Sigma^{-1}\Sigma\Sigma^{-1}\mu}{B^2}=\frac{C}{B^2} \tag{7}$$

4. Compute $\mu_d-\mu_g$ as a function of A, B, C, and Δ. Under which condition $\mu_d-\mu_g>0$?
Solution

$$\mu_d-\mu_g=\frac{C}{B}-\frac{B}{A}=\frac{\Delta}{AB} \tag{8}$$

We have $\mu_d-\mu_g>0$ whenever $B>0$.

5. Plot a diagram in the mean - standard deviation space with:

   (a) the minimum-variance set

   (b) the mean-variance *efficient* set

   (c) the global minimum variance portfolio

   (d) the w_d portfolio. Under which condition will w_d lie on the upper limb of the hyperbola?

   Solution
   See class notes for the diagram. The portfolio w_d will lie on the upper limb of the hyperbola whenever $\mu_d-\mu_g>0$, that is, whenever $B>0$.

6. The two-fund separation result can be useful for computing the covariance between two minimum-variance portfolios $w_a=(1-a)\,w_g+aw_d$ and $w_b=(1-b)\,w_g+bw_d$. Compute the covariance between these two portfolios by starting with the formula $Cov\left[\tilde{R}_a,\tilde{R}_b\right]=w_a'\Sigma w_b$. Your final result should contain only a, b, A, B, C, and Δ.

Solution

$$
\begin{aligned}
w_a'\Sigma w_b &= \left((1-a)\,w_g' + aw_d'\right)\Sigma\left((1-b)\,w_g + bw_d\right) \\
&= (1-a)(1-b)\,\sigma_g^2 + \left[(1-a)\,b + a\,(1-b)\right]\sigma_{dg} + ab\sigma_d^2 \\
&= \frac{1}{A} + \frac{ab\Delta}{AB^2} \qquad (9)
\end{aligned}
$$

7. Obtain directly by using the previous formula the covariance between any minimum variance portfolio and the global minimum variance portfolio. Does it depend on a or b?
   Solution
   One of the two portfolios from the previous exercise (let's say w_a) is the global minimum variance portfolio. Thus, $a = 0$. Therefore, the covariance between any minimum variance portfolio and the global minimum variance portfolio is $1/A$.

Problem 3 (Portfolio Theory, 25 points)

Consider an economy with 3 risky assets with expected returns

$$
\mu = \begin{bmatrix} 0.13 \\ 0.12 \\ 0.10 \end{bmatrix} \qquad (10)
$$

The variance-covariance matrix of returns is given by

$$
\Sigma = \begin{bmatrix} 0.016 & 0.007 & 0.005 \\ 0.007 & 0.02 & 0.003 \\ 0.005 & 0.003 & 0.01 \end{bmatrix} \qquad (11)
$$

The inverse of the variance-covariance matrix of returns is given by

$$
\Sigma^{-1} = \begin{bmatrix} 83.91916 & -24.165202 & -34.710018 \\ -24.165202 & 59.314587 & -5.711775 \\ -34.710018 & -5.711775 & 119.068541 \end{bmatrix} \qquad (12)
$$

The risk-free rate is 5%.

1. What is the optimal portfolio for an investor (call him X) with a risk aversion (a_X) of 5? Does the investor lend or borrow? How much?
   Solution
   With a riskless asset that pays $R = 5\%$, the optimal portfolio for an investor with a risk aversion of $a_X = 5$ is

$$
w = \frac{1}{a_X}\Sigma^{-1}\left(\mu - R\mathbf{1}\right) = \begin{bmatrix} 0.6573 \\ 0.3866 \\ 0.5554 \end{bmatrix} \qquad (13)
$$

   The riskless asset position is $\omega_0 = 1 - w'1 = 1.5993$. Thus, the investor borrows 59.93% of his wealth in order to take a levered position in the tangency portfolio.

2. What is the composition of the market portfolio? Its expected return and standard deviation of returns?
   Solution
   The market portfolio is the investor's optimal portfolio, normalized so that the sum of the weights equals 1,

$$w_M = \begin{bmatrix} 0.4110 \\ 0.2418 \\ 0.3473 \end{bmatrix} \tag{14}$$

   Its expected return is $\mu_M = 0.1172$*, its standard deviation* $\sigma_M = 0.0916$*.*

3. How much is the economy-wide aggregate risk aversion implicit in the market portfolio? Interpret this value.
   Solution
   The economy-wide aggregate risk aversion implicit in the market portfolio is given by

$$a_M = 5 \cdot 1.5993 = 7.9965 \tag{15}$$

   This value means that assets are priced as if all investors had a risk aversion coefficient of 7.9965. Investors with a risk aversion coefficient above (below) this value will be lenders (borrowers).

4. Consider now a second investor (call him Y) who has *the same initial wealth* as the investor X. Let's suppose that there are only two investors in this market and that the risk-free asset is in *zero net supply.*

   (a) what is the position in the risk free asset of investor Y? Interpret your result.
   (b) what is the optimal portfolio of investor Y?
   (c) compute the risk aversion of investor Y (a_Y).
   (d) can you find a relationship between the economy-wide aggregate risk aversion and the individual risk aversions a_Y and a_Y ? Explain intuitively your result.

   Solution
   a) Investor X borrows 59.93% of his wealth, and the risk-free asset is in zero net supply. Thus, investor Y will lend 59.93% of his wealth, in order to obtain the equilibrium.
   b) The fraction of wealth invested in the market portfolio for investor Y will be $1 - 0.5993 = 0.4007$*. Thus, the optimal portfolio for investor Y will be*

$$w_Y = \begin{bmatrix} 0.1647 \\ 0.0969 \\ 0.1391 \end{bmatrix} \tag{16}$$

   c) We know that

$$a_M = a_Y \cdot 0.4007 \tag{17}$$

   Thus, the risk aversion of investor Y is $a_Y = 19.9561$*.*
   d) Investor X has a position of 1.5993 in the market portfolio, while investor Y has a position of 0.4007 in the same portfolio. Thus, the total

market portfolio is $1.5993 + 0.4007 = 2$. *Making an weighted average of the risk aversions of the two investors yields*

$$\frac{1.5993}{2}a_X + \frac{0.4007}{2}a_Y = 7.9965 = a_M \tag{18}$$

Interpretation: when we average the risk aversion of all the individuals in the economy, using weights corresponding to their position in the market portfolio, we obtain the economy-wide aggregate risk-aversion.

Problem 4 (CAPM, 15 points)

Consider a financial market with 3 assets, X, Y and Z. Firm X accounts for 0.5 of the total market capitalization, firm Y accounts for 0.3 of the total market capitalization, and firm Z accounts for 0.2 of the total market capitalization. Following analyst estimates, the expected return for X is 10%, for Y is 15%, and for Z is 20%. Standard deviations of returns are estimated at 15%, 20% and 30% respectively. Finally, the correlation matrix of the returns of X, Y, and Z is

$$C = \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{19}$$

1. Calculate the expected return of the market portfolio.
   Solution
   The expected return of the market portfolio is 13.5%.

2. Calculate the standard deviation of returns for the market portfolio *(using matrix notation might help).*
   Solution
   The variance-covariance matrix of returns is

$$\Sigma = \begin{bmatrix} 0.0225 & 0.009 & 0 \\ 0.009 & 0.04 & 0 \\ 0 & 0 & 0.09 \end{bmatrix} \tag{20}$$

   Thus, the standard deviation of returns for the market portfolio is 12.46%.

3. Compute the beta of X (β_X) and the beta of Y (β_Y) with respect to the market portfolio.
   Solution

$$\beta_X = \frac{Cov\left[r_X,\ r_M\right]}{\sigma_M^2} = \frac{Cov\left[r_X,\ 0.5r_X + 0.3r_Y + 0.2r_Z\right]}{\sigma_M^2} = \frac{0.5\sigma_X^2 + 0.3\sigma_{XY}}{\sigma_M^2} = 0.8986 \tag{21}$$

$$\beta_Y = \frac{Cov\left[r_Y,\ r_M\right]}{\sigma_M^2} = \frac{Cov\left[r_Y,\ 0.5r_X + 0.3r_Y + 0.2r_Z\right]}{\sigma_M^2} = \frac{0.5\sigma_{XY} + 0.3\sigma_Y^2}{\sigma_M^2} = 1.0628 \tag{22}$$

4. Use the previous results to find the beta of Z (β_Z).
   Solution
   We know that

$$0.5\beta_X + 0.3\beta_Y + 0.2\beta_Z = 1 \tag{23}$$

It follows that

$$\beta_Z = \frac{1 - 0.5\beta_X - 0.3\beta_Y}{0.2} = 1.1594 \tag{24}$$

5. Verify rapidly if a risk-free rate of 5% is valid for this economy.
   Solution
   Let's verify the CAPM for asset X

$$\begin{aligned} \mu_X - r_f &= 0.1 - 0.05 = 0.05 & (25) \\ \beta_X \left(\mu_M - r_f\right) &= 0.8986\,(0.135 - 0.05) = 0.0764 & (26) \end{aligned}$$

   The numbers are different, so CAPM cannot hold with a risk-free rate of 5%.