Investments

Session 5. Arbitrage Pricing Theory

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Outline

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Novel Feature

- Arbitrage pricing theory is a new and different approach to determining asset prices. It is based on the law of one price: *two items that are the same can't sell at different prices.*
- The CAPM predicts than only one type of nondiversifiable risk influences expected security returns: market risk i.e. covariance with the market portfolio.
- On the other hand, the APT accepts a variety of different risk sources, such as the business cycle, interest rates and inflation.
- Like the CAPM, the APT distinguishes between idiosyncratic and systematic sources of risk. However, in the APT there are several sources of systematic risk. The expected return on an asset is driven by its exposure to the different sources of systematic risk.

- The APT requires that te returns on any stock be linearly related to a set of indices (or factors) as shown below:

$$R_j = a_j + b_{j1} I_1 + b_{j2} I_2 + ... + b_{jK} I_K + \varepsilon_j \tag{1}$$

where

  - $a_j =$ the expected level of return for stock j if all indices have a value of zero
  - $I_k =$ the value of the kth index that impacts the return on stock j
  - $b_{jk} =$ the sensitivity of stock j's return to the kth index
  - $\varepsilon_j =$ a random error term with mean zero and variance equal to $\sigma^2_{\varepsilon j}$.

- Some assumptions are required to fully describe the process-generating security returns:

$$\begin{aligned} E\left[\varepsilon_i \varepsilon_j\right] &= 0, \quad \text{for all } i \text{ and } j \text{where } i \neq j \\ E\left[e_j\left(I_k - \bar{I}_k\right)\right] &= 0, \quad \text{for all stocks and indices} \end{aligned}$$

with $\bar{I}_k = E\left[I_k\right]$.

- Taking the expected value of equation (1) and substracting it from equation (1), we have

$$R_j - \mu_j = b_{j1} f_1 + \ldots + b_{jK} f_K + \varepsilon_j \tag{2}$$

where $f_k = I_k - \bar{I}_k$. The difference between the realized return and the expected return for any asset is

1. the sum, over all risk factors k, of the asset's risk exposure (b_{jk}) to that factor, multiplied by the realization for that risk factor, f_k,
2. plus an asset-specific idiosyncratic error term, ε_j.

- Note that the risk factors themselves may be correlated, as may the asset-specific shocks for different assets.
- To derive the APT, one postulates that pure arbitrage profits are impossible.

The Notion of Arbitrage

- The principal strength of the APT approach is that it is based on the no arbitrage condition.
- Intuitively, arbitrage means “there is no such thing as a free lunch”. Two assets with identical attributes should sell for the same price, and so should an identical asset trading in two different markets (Law of one Price).
- Arbitrage is a common feature of competitive markets. Even tourists ignorants of the theory of finance can turn into arbitrageurs (exchange rate example).
- Arbitrage has been elevated to the level of a driving force by Modigliani and Miller in 1958. They used the arbitrage argument to prove that the value of a firm as a whole is independent of its capital structure (MM theorems).

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- To derive the APT equilibrium pricing relationship, one constructs a portfolio w with the following properties:

$$w'\mathbf{1} = \sum_{n=1}^{N} \omega_n = 0 \quad (3)$$

$$w'b_k = \sum_{n=1}^{N} \omega_n b_{nk} = 0 \quad \forall k \quad (4)$$

$$w'\varepsilon = \sum_{n=1}^{N} \omega_n \varepsilon_n \approx 0 \quad (5)$$

These conditions can be met if there is a sufficient number of securities available on the market.

- Since the portfolio has zero initial cost and a risk of zero, it must have an expected return of zero, i.e. we must have

$$w'\mu = \sum_{n=1}^{N} \omega_n \mu_n = 0 \tag{6}$$

- In other words: if w is orthogonal to a vector of ones (the first condition) and orthogonal to K vectors of b_k's, then it is also orthogonal to the vector of expected returns, μ.
- There is a theorem in linear algebra that states the following: if the fact that a vector is orthogonal to $M-1$ vectors implies that it is also orthogonal to the Mth vector, then the Mth vector can be expressed as a linear combination of the $M-1$ vectors.

- Here is some intuition for why this result holds. Write μ as some linear combination of 1 and the b_k's, plus an error term d (which will be nonzero if μ cannot be written as a linear combination of the vectors),

$$\mu = P_0\mathbf{1} + \sum_{k=1}^{K} P_k b_k + d \tag{7}$$

Then,

$$w'\mu = P_0 w'\mathbf{1} + \sum_{k=1}^{K} P_k w' b_k + w'd = w'd \tag{8}$$

- Note that none of the restrictions on w imposed by the constraints that $w'\mathbf{1} = 0$ and $w'b_k = 0$ allow us to say anything about the remaining term $w'd$. Therefore, the only way to ensure that $w'd = 0$ so that $w'\mu = 0$ is to set $d = 0$, which says that μ can indeed be written as a linear combination of $\mathbf{1}$ and the b_k's.

- We have just shown that for there to be no arbitrage, one must be able to write μ as a linear combination of 1 and the K vectors b_k,

$$\mu = P_0 \mathbf{1} + \sum_{k=1}^{K} P_k b_k \tag{9}$$

- This is the main APT theorem: under the above assumptions, there exist $K+1$ numbers $P_0, P_1, ..., P_K$, not all zero, such that the expected return on asset j is approximately equal to

$$\mu_j \approx P_0 + b_{j1} P_1 + ... + b_{jK} P_K \tag{10}$$

- Under the additional assumptions that (i) there exists a portfolio with no nonsystematic risk and that (ii) some investor considers it his optimal portfolio, one can show that the above equation holds with equality (Chen and Ingersoll, JF 1983),

$$\mu_j = P_0 + b_{j1} P_1 + ... + b_{jK} P_K \tag{11}$$

- P_k is the price of risk (the risk premim) for the kth risk factor, and determines the risk-return tradeoff.
- Consider a portfolio p that is perfectly diversified ($\varepsilon_p = 0$) and with no factor exposures $\left(b_{pk} = 0,\ \forall k\right)$. Such portfolio has zero risk, and its expected return is P_0. Therefore, P_0 must equal the risk-free rate of return R.
- Similarly, the risk premium for the kth risk factor, P_k, is the return, in excess of the risk-free rate, earned on an asset that has an exposure of $b_{jk} = 1$ to the kth factor and zero risk exposure to all other factors $\left(b_{jh} = 0,\ \forall h \neq k\right)$:

$$P_k = \mu_{F,k} - R \tag{12}$$

- Thus P_k are returns for bearing the risks associated with the indices I_k, or factor risk premiums.

- Substituting the expected return relationship (11) into the multi-factor model specification (2) yields

$$\begin{aligned} R_j - \mu_j &= R_j - (P_0 + b_{j1}P_1 + \ldots + b_{jK}P_K) \\ &= b_{j1}f_1 + \ldots + b_{jK}f_K + \varepsilon_j \end{aligned} \tag{13}$$

which, using $P_0 = R$, can be rewritten to yield the full APT equation

$$R_j - R = b_{j1}(P_1 + f_1) + \ldots + b_{jK}(P_K + f_K) + \varepsilon_j \tag{14}$$

- This says that the realized return on an asset in excess of the risk-free rate is the sum of 3 components
  1. expected macroeconomic factor return (the P's), i.e. the reward for the risks taken,
  2. unexpected macroeconomic factor return (the f's), and
  3. an idiosyncratic component (ε).

Interpretation: Several Risk Factors

- Taking expectations on the full APT equation yields

$$\mu_j - R = b_{j1}P_1 + ... + b_{jK}P_K \tag{15}$$

which says that the expected excess return on an asset is the sum over all factors k of the product of the factor's risk premium P_k and of the asset's risk exposure to that factor b_{jk}.

- As the exposure of a portfolio to a particular factor k is increased, the expected return on the portfolio is increased if $P_k > 0$.
- Note that the big difference between the CAPM and the APT is that the CAPM postulates that the risk premium on an asset depends on a single factor: covariance with the market portfolio. In the APT, on the other hand, several factors may drive expected returns *(but the APT does not say what they are)*.

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Equivalence

- We should discuss the fact that the APT model and, in fact, the existence of a multifactor model, is not necessarily inconsistent with the CAPM.
- The APT equation is

$$\mu_j = R + \sum_{k=1}^{K} b_{jk} P_k \tag{16}$$

- Recall that if the CAPM is the equilibrium model, it holds for all securities, as well as all portfolios of securities. We have seen that P_k is the excess return on a portfolio with a b_{jk} of one on one index and b_{jk} of zero on all other indices, thus the indices can be represented by portfolios of securities.
- If the CAPM holds, the equilibrium return on each P_k is given by

$$P_k = \mu_{F,k} - R = \beta_{F,k} (\mu_M - R), \quad \forall k \tag{17}$$

Equivalence (cont.)

- Substituting into equation (16) yields

$$\mu_j - R = (\mu_M - R)\left(\sum_{k=1}^{K} b_{jk}\beta_{F,k}\right) \quad (18)$$

- Defining β_j as $\sum_{k=1}^{K} b_{jk}\beta_{F,k}$ results in the expected return of security j being priced by the CAPM

$$\mu_j = R + \beta_j(\mu_M - R) \quad (19)$$

- The APT solution with multiple factors *appropriately priced* is fully consistent with the CAPM. Conversely, if the APT is true and the K restrictions on the P_k's hold, then the CAPM is also true.

Equivalence (cont.)

- This result is important for empirical testing: employing statistical techniques to estimate the P_k's and finding that more than one coefficient is significantly different from zero is not sufficient proof to reject any CAPM. If the P_k's are not significantly different form $\beta_{F,k}(\mu_M - R)$, the empirical results could be fully consistent with the CAPM.
- Thus, it is perfectly possible that more than one index explains the covariance between security returns but that the CAPM holds.
- However, in empirical tests, the restrictions on APT coefficients imposed by the CAPM are rejected.

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- The proof of any economic theory is how well it describes reality. Let us review the structure of APT that will enter any test procedure.
- We can write the multifactor return-generating process as

$$R_j = a_j + \sum_{k=1}^{K} b_{jk} I_k + \varepsilon_j \tag{20}$$

- The APT model that arises from this return-generating process can be written as

$$\mu_j = R + \sum_{k=1}^{K} b_{jk} P_k \tag{21}$$

- Notice from equation (20) that each security j has a unique sensitivity to each I_k but that any I_k has a value that is the same for all securities. Any I_k affects more than one security (if it did not, it would have been compounded in the residual term ε_j).

- These I_k's have generally been given the name factors in the APT literature. The factors affect the returns of more than one security and are the sources of covariance between securities.
- The b_{jk}'s are unique to each security and represent attributes or a characteristics of the security.
- Finally, from equation (21) we see that P_k is the extra expected return required because of a security's sensitivity to the kth attribute.
- Recall that the APT does not say what the K factors are. Therefore, in order to test the APT, one must test equation (21), which means that one must have estimates of the b_{jk}'s. However, to estimate the b_{jk}'s we must have definitions of the relevant I_k's.

- Three approaches can be used to estimate and test the APT:
  1. The most general approach is to *use statistical techniques and estimate simultaneously factors (I_k's) and firm attributes (b_{jk}'s)*. The results thus obtained have the drawback that the estimated factors are difficult to interpret because they are non-unique linear combinations of more fundamental underlying economic forces.
  2. *Specify a set of characteristics (b_{jk}'s) a priori.* Then the values of the P_k's would be estimated via regression analysis.
  3. The drawback of the first two methods is that they use stock returns to explain stock returns. A third approach would be to *use economic theory and knowledge of financial markets to specify K risk factors* that can be measured from available macroeconomic and financial data. This is the preferred approach.

In Practice

- Let us take a look at the third approach. This part is following Burmeister, Roll and Ross, (2003) "Using Macroeconomic Factors to Control Portfolio Risk". The factors should
  - be easy to interpret,
  - be robust over time, and
  - explain as much as possible of the variation in stock returns.
- Empirical research has established that one set of five factors meeting these criteria is the following:

In Practice (cont.)

Name	Measure	Risk Premium
Confidence Risk: Investors' willingness to take risks	Rate of return on risky corporate bonds minus rate of return on government bonds (positive values means increased investor confidence)	$P_1 = 2.59\%$, $b_{j1} > 0$
Time Horizon Risk: change in investors' desired time to payouts	Return on 20-year government bonds minus return on 30-day Treasury bills ($f_2 > 0$ when the price of long-term bonds rises relative to the t-bill price)	$P_2 = -0.66\%$, $b_{j2} > 0$
Inflation Risk: unexpected inflation	Inflation surprise: actual inflation minus expected inflation	$P_3 = -4.32\%$, usually $b_{j3} < 0$
Business Cycle Risk: unanticipated changes in real business activity	Change in the index of business activity (i.e. value in month $T+1$ minus value in month T	$P_4 = 1.49\%$, usually $b_{j4} > 0$
Market-Timing Risk	Part of the S&P 500 total return that is not explained by the first four factors plus a constant	$P_5 = 3.61\%$, $b_{J5} > 0$

In Practice (cont.)

- For any asset or portfolio, we therefore have

$$\mu_j - R = 2.59b_{j1} - 0.66b_{j2} - 4.32b_{j3} + 1.49b_{j4} + 3.61b_{j5} \tag{22}$$

- This says that the risk premium on any asset or portfolio is the sum of the product, over all K risk factors, of the asset's exposure and of the corresponding price of risk.
- As an example, for the S&P 500, the exposures are $b_1 = 0.27$, $b_2 = 0.56$, $b_3 = -0.37$, $b_4 = 1.71$, $b_5 = 1$.
- Therefore, using the factor risk premia, the expected excess return on the S&P is

$$\mu_M - R = 8.09\% \tag{23}$$

- Computing the expected return on some assets or portfolios is only one of the many uses of the APT. We now briefly discuss the others.

Tilting & Other Strategies

- Determining Risk Exposure: Using the APT, one can determine the exposure of one's portfolio to the different factors. Let w denote the vector of portfolio weights and

$$B = \begin{bmatrix} b_{11} & \dots & b_{N1} \\ \dots & & \\ b_{1K} & \dots & b_{NK} \end{bmatrix} \qquad (24)$$

the (stacked) matrix of factor exposures of the N assets. Then, the (column) vector of the portfolio's factor exposures is given by

$$b_p = B \cdot w \qquad (25)$$

The exposure of the portfolio to the kth factor is simply a weighted average of the individual assets' exposure to that factor.

Tilting & Other Strategies (cont.)

- Tilting (Making a Factor Bet): If you consider that you have superior knowledge about the future evolution of some of the factors, you can increase the exposure of your portfolio to the factors that are expected to lead to improvements in returns and reduce the exposure to those factors that are expected to lead to a deterioration in returns. To do so, construct factor portfolios with an exposure of 1 to the kth factor and 0 to all other factors. Let $\mathbf{1_k}$denote this target exposure pattern. Then, the portfolio weights must solve

$$B \cdot w_k = \mathbf{1_k} \tag{26}$$

  If the number of assets is equal to the number of factors, then one has $w_k = B^{-1} 1_k$. If there are more assets than factors, then there will be an inifinite number of factor portfolios solving $B \cdot w_k = 1_k$.

- In order to find factor portfolios in this more general case, we can use the following results from linear algebra:

Tilting & Other Strategies (cont.)

- An $n \times m$ matrix X is the pseudoinverse of an $m \times n$ matrix A if the following four conditions hold: $AXA = A$, $XAX = X$, $(AX)' = AX$, and $(XA)' = XA$. We will denote the pseudoinverse of a matrix A by A^+. The matlab command for the pseudoinverse is "pinv".
- A necessary and sufficient condition for the vector equation $Ax = b$ to have a solution is that $AA^+b = b$, in which case the general solution is

$$x = A^+b + \left(I - A^+A\right)q \tag{27}$$

where q is an arbitrary vector (see, for example, Magnus and Neudecker, *Matrix Differential Calculs with Applications to Statistics and Econometrics*, Chapter 2, Theorem 12).

- Hence, the set of factor portfolios for the kth factor is given by

$$w_k = B^+1_k + \left(I - B^+B\right)q \tag{28}$$

Tilting & Other Strategies (cont.)

- Similarly, the set of portfolios with a factor exposure of b_p is

$$w = B^{+} b_p + \left(I - B^{+} B \right) q \tag{29}$$

- If one is looking for the portfolio that has minimum risk subject to meeting the target factor exposure, one needs to solve

$$\min_q w' \Sigma w \tag{30}$$

  with w given by (29).

- Long-Short Strategies: If one has stock selection skills but no macroeconomic prediction skills, one can still use the APT, as follows:
  1. buy stocks with high expected idiosyncratic return ε,
  2. short stocks with negative expected idiosyncratic return.

Tilting & Other Strategies (cont.)

If the long and the short portfolio are constructed so as to have opposite exposures to each of the risk factors, the systematic risk will be zero and expected return will lie above the risk-ree rate. The APT helps ensure that the portfolios are appropriately constructed.

- Return Attribution: After observing the returns on one's portfolio, one can determine their source:
  1. expected macroeconomic factor return (the P's), i.e. the reward for the risks taken,
  2. unexpected macroeconomic factor return (the f's), and
  3. anything that remains (ε), which one can attribue to luck or to stock selection.

Outline

Summary

- Like the CAPM, the basic concept of the APT is that differences in expected return must be driven by differences in non-diversifiable risk.
- The APT is based purely on no-arbitrage condition. It is not an equilibrium concept, and does not depend on having a market portfolio.
- Through the use of arbitrage, APT provides investors with strategies for betting on their forecasts of the factors that shape stock returns.
- The construction of APT enables it to avoid the rigid and often unrealistic assumptions required by CAPM.
- CAPM specifies where asset prices will settle, given investor preferences, but it is silent about what produces the returns that investors expect. It also identifies only one factor as the dominant influence on stock returns.

Summary (cont.)

- APT fills those gaps by providing a method to measure how stock prices will respond to changes in the multitude of economic factors that influence them, such as economic growth, inflation , interest rate patterns, etc.
- The CAPM assumes an unobservable "market" portfolio. The APT is based on the assumption of no arbitrage profits in well-diversified portfolios.
- The APT provides no guidance for identification of the various market factors and appropriate risk premiums for these factors.

For Further Reading

1. Roll and Ross, *The Arbitrage Pricing Theory Approach to Strategic Portfolio Planning*, Financial Analysts Journal 1984.
   - intuitive description of APT and a discussion ot its merits for portfolio management.
2. Burmeister, Roll and Ross, *Using Macroeconomic Factors to Control Portfolio Risk*, 2003.
   - understanding the macroeconomic forces impacting stock returns.

Formula Sheet

- The multifactor return-generating process

$$R_j = a_j + \sum_{k=1}^{K} b_{jk} I_k + \varepsilon_j \tag{31}$$

- The APT model that arises from this return-generating process

$$\mu_j = R + \sum_{k=1}^{K} b_{jk} P_k \tag{32}$$