Investments

Session 4. The Capital Asset Pricing Model

EPFL - Master in Financial Engineering
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Outline

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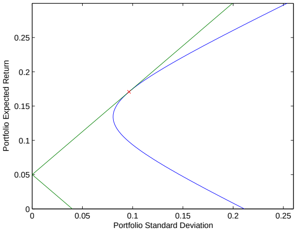
Previous Results

- We have seen that the Capital Market Line (the line that goes through the riskless asset and the tangency portfolio) gives the best risk-return tradeoff available to investors in a mean-variance world.
- As a result, all investors' optimal portfolio consisted in a combination of the riskless asset and the tangency portfolio. We know now that portfolio choice could be made in two steps:
  1. Selection of the optimal risky portfolio (the tangency portfolio).
  2. Based on individual risk tolerance, choice of the optimal combination between the riskless asset and the investment in the tangency portfolio.
- The question we whish to answer now is: what does this "separation" imply for capital asset prices in equilibrium?

A Nobel Prize idea: William Sharpe

- Sharpe was the first to answer this question and developed the capital asset pricing model. He shared the 1990 Nobel Prize in economics with Harry Markowitz.
  "I said what if everyone was optimizing? They've all got their copies of Markowitz and they're doing what he says. Then some people decide they want to hold more IBM, but there aren't enough shares to satisfy demand. So they put price pressure on IBM and up it goes, at which point they have to change their estimates of risk and return, because now they're paying more for the stock. That process of upward and downward pressure on prices continues until prices reach an equilibrium and everyone collectively wants to hold what's available. At that point, what can you say about the relationship between risk and return? The answer is that expected return is proportionate to beta relative to the market."
  Source: J. Burton (1998), "Revisiting the CAPM", *Dow Jones Asset Manager*, 1998.

- In a mean-variance world, all investors optimally hold a combination of the riskless asset and of the tangency portfolio.
- Suppose an investor chooses to hold another combination (i.e., to bear idiosyncratic risk). Will he earn a higher return?
- Just looking at the graph, we can see that the answer is no. The market pays no risk premium for being inside the frontier.



- Using the CML, the expected return on any efficient portfolio w_e (the ones on the CML) is given by

$$\mu_e = R + \frac{\mu_t - R}{\sigma_t}\sigma_e \quad (1)$$

The first term (R) is compensation for the time value of money, the second term $\frac{\mu_t - R}{\sigma_t}\sigma_e$ is compensation for risk.

- This second term is the product of the *market price of risk*,

$$\lambda = \frac{\mu_t - R}{\sigma_t} \quad (2)$$

and of the amount of risk borne by the investor, σ_e. The market price of risk λ tells us how much the market compensates investors for each unit of risk. The risk premium $\frac{\mu_t - R}{\sigma_t}\sigma_e$ is therefore linear in the amount of systematic risk borne by the investor.

- Now, recall that as claimed earlier, only systematic risk is is compensated. Bearing idiosyncratic risk does not entitle to a risk premium.
- Consider an asset or portfolio with total risk σ_j. Letting ρ_{jt} denote the correlation of its returns with the tangency portfolio, the risk σ_j can be composed into
  1. systematic risk $\rho_{jt}\sigma_j$ and
  2. idiosyncratic risk $\sqrt{1-\rho_{jt}^2}\sigma_j$.
- Therefore, using the market price for risk λ, the risk premium on the asset must be

$$\begin{aligned} \mu_j - R &= \lambda \rho_{jt}\sigma_j = \frac{\mu_t - R}{\sigma_t}\rho_{jt}\sigma_j = (\mu_t - R)\,\rho_{jt}\frac{\sigma_j}{\sigma_t} \\ &= (\mu_t - R)\frac{Cov\left(\tilde{R}_j, \tilde{R}_t\right)}{Var\left(\tilde{R}_t\right)} \end{aligned} \tag{3}$$

The CAPM relationship

- In this section we will derive the CAPM more rigorously. To begin with, make the following assumptions:
    1. Individual investors cannot affect prices. That is, there are many buyers and sellers (perfect competition).
    2. All investors plan for one identical holding period and they are mean-variance optimizers. That is, they choose the portfolio w_i that maximizes $\mu - \frac{a_i}{2}\sigma^2$, where a_i denotes investor i's degree of risk aversion (any other mean-variance preferences would work as well).
    3. There is a riskless asset with return R and N risky assets with expected return μ and variance-covariance matrix Σ.
    4. Investors pay neither taxes on returns nor transactions costs.
    5. Each asset is perfectly divisible.
    6. The riskless asset can be bought or sold in unlimited amounts.
    7. All investors have homogenous expectations about μ and Σ.

The CAPM relationship (cont.)

- Under these assumption, each investor's portfolio is the solution to

$$\max_{w_i} \left(R + w_i' (\mu - R\mathbf{1}) - \frac{a_i}{2} w_i' \Sigma w_i \right) \tag{4}$$

which yields the first-order condition (this becomes standard by now)

$$\mu - R\mathbf{1} = a_i \Sigma w_i \tag{5}$$

and the optimal portfolio

$$w_i = \frac{1}{a_i} \Sigma^{-1} (\mu - R\mathbf{1}) \tag{6}$$

The CAPM relationship (cont.)

- Now, sum over all investors, weighting each investor by his relative wealth $\frac{W_i}{\sum_{i=1}^{I} W_i} = \frac{W_i}{W}$ to get

$$\sum_{i=1}^{I} \left(\frac{W_i}{W} w_i \right) = \sum_{i=1}^{I} \left(\frac{W_i}{W} \frac{1}{a_i} \right) \Sigma^{-1} (\mu - R\mathbf{1}) \tag{7}$$

- Since the riskless asset is in zero net supply, the left hand side of this expression must be the tangency portfolio, or the market portfolio. Therefore, we can write

$$w_t = w_M = \sum_{i=1}^{I} \left(\frac{W_i}{W} \frac{1}{a_i} \right) \Sigma^{-1} (\mu - R\mathbf{1}) \tag{8}$$

The CAPM relationship (cont.)

or, solving for $\mu - R\mathbf{1}$,

$$\mu - R1 = \frac{1}{\sum_{i=1}^{I}\left(\frac{W_i}{W}\frac{1}{a_i}\right)}\Sigma w_M \tag{9}$$

- The problem we face here is that we do not know $\sum_{i=1}^{I}\left(\frac{W_i}{W}\frac{1}{a_i}\right)$. To find it, pre-multiply the above expression by w_M' to get

$$w_M'(\mu - R\mathbf{1}) = \frac{1}{\sum_{i=1}^{I}\left(\frac{W_i}{W}\frac{1}{a_i}\right)} w_M'\Sigma w_M \tag{10}$$

The CAPM relationship (cont.)

- Using $w_M'\mathbf{1} = 1$, rewrite this expression as

$$\mu_M - R = \frac{1}{\sum_{i=1}^{I}\left(\frac{W_i}{W}\frac{1}{a_i}\right)}\sigma_M^2 \tag{11}$$

to get

$$\frac{1}{\sum_{i=1}^{I}\left(\frac{W_i}{W}\frac{1}{a_i}\right)} = \frac{\mu_M - R}{\sigma_M^2} \tag{12}$$

The CAPM relationship (cont.)

- Then, substitute into $\mu - R\mathbf{1} = \frac{1}{\sum_{i=1}^{I}\left(\frac{W_i}{W}\frac{1}{a_i}\right)}\Sigma w_M$ to obtain

$$\mu - R\mathbf{1} = \frac{\mu_M - R}{\sigma_M^2}\Sigma w_M \tag{13}$$

Finally, note that $\Sigma w_M = Cov\left(\tilde{R}, \tilde{R_M}\right)$ and write the CAPM relationship as

$$\mu - R1 = \frac{Cov\left(\tilde{R}, \tilde{R_M}\right)}{Var\left(\tilde{R_M}\right)}(\mu_M - R) \tag{14}$$

Outline

Interpretation

- The CAPM relationship says that the equilibrium risk premium on any asset j, $\mu_j - R$, equals

$$\mu_j - R = \beta_j \left(\mu_M - R\right), \quad \beta_j = \frac{Cov\left(\tilde{R}_j, \tilde{R_M}\right)}{Var\left(\tilde{R_M}\right)} = \rho_{jM} \frac{\sigma_j}{\sigma_M} \tag{15}$$

- Thus, the market only compensates investors for bearing *systematic* (i.e., market) risk, not idiosyncratic risk. There is a single source of systematic risk: covariance with the market portfolio. The expected return on an asset with returns uncorrelated with those of the market portfolio is R, even if the asset is risky.

Interpretation (cont.)

- The relationship between the expected return on an asset and its β is linear.
- The risk premium on an asset is proportional to the excess return on the market portfolio, $\mu_M - R$.
- What the CAPM achieves is to move from the *capital market line* (CML) to the *security market line* (SML):

$$\mu_j = R + \beta_j (\mu_M - R) \tag{16}$$

In equilibrium, all assets plot on the security market line.

Interpretation (cont.)

- The CML versus the SML.
  - The CML graphs the risk premiums of efficient portfolios (complete portfolios of risky securities and the risk-free asset) as a function of portfolio standard deviation.
  - The SML graphs the risk premium of individual assets, thus measuring the risk contribution that an asset would make to the standard deviation of a portfolio. This contribution is measured by the asset's beta.
  - "Fairly priced" assets lie on the SML; *alpha* measures the distance between the fair and the actually expected return on the asset. Assets plotting below the SML are overpriced; those plotting above the SML are underpriced.

Beta and expected returns

- Why should a positive dependence between β and the expected return on an asset arise?
- Consider an asset with a high β. It is procyclical: whenever the market does well, it tends to do very well. But if the market does well, then overall wealth is high, and the marginal utility of consumption is low. In a sense, this asset pays off a lot when money is least valuable, and very little when money is very valuable. Therefore its price will be low and its expected return high.
- On the other hand, an asset with negative β is anticyclical: it pays off a lot when aggregate wealth is low (and money very valuable), and little when overall wealth is high (and money less valuable). As a result, its price will be high, and expected return low (an example of such an asset is gold, which has a correlation of -0.4 with stocks).

Outline

Extensions of the CAPM

- In the above analysis, we made a number of restrictive assumptions, such as
  1. No short-selling constraints,
  2. Existence of a riskless asset allowing unlimited lending and borrowing at the same rate,
  3. No taxes,
  4. All assets can be marketed,
  5. Heterogeneous expectations,
  6. Non-price-taking behavior,
  7. Single investment period.
- We will now consider how robust the CAPM equation is to relaxation of these assumptions. Only short-selling constraints and the no riskless asset case will be discussed here.

Short-Selling Constraints

- What is the effect of short-selling constraints on equilibrum asset prices?
- Recall that in the CAPM, all investors hold the market portfolio in equilibrium. Therefore, in equilibrium, no investor sells any security short.
- As a result, the short-selling constraints is non-binding and equilibrium prices are unaffected by it. The standard version of the CAPM holds here as well.

The Zero-Beta CAPM

- When investors can no longer borrow or lend at a common risk-free rate, they may choose risky portfolios from the entire set of efficient frontier portfolios, according to how much risk they choose to bear.
- An equilibrium expected return-beta relationship in the case of restricted risk-free investments has been developed by Fischer Black. The model is based on the three following properties of mean-variance efficient portfolios:
  1. any portfolio constructed by combining efficient portfolios is on the efficient frontier.
  2. every portfolio on the efficient frontier has a "companion" portfolio on the inefficient portion of the minimum-variance frontier with which it is uncorrelated.
  3. the expected return of any asset can be expressed as an exact, linear function of the expected return on any two frontier portfolios.

The Zero-Beta CAPM (cont.)

- We assume that the risky assets can be sold short. The result we derive below also obtains if we allow lending and borrowing, but at different rates.
- If there is no riskless asset, each investor solves

$$\max_{w_i}\left(w_i'\mu - \frac{a_i}{2} w_i'\Sigma w_i\right) \quad s.t. \quad w_i'\mathbf{1} = 1 \tag{17}$$

  yielding the first-order condition $\mu - a_i \Sigma w_i - \lambda \mathbf{1} = 0$ and the optimal portfolio

$$w_i = \frac{1}{a_i}\Sigma^{-1}(\mu - \lambda\mathbf{1}) = b_i w_d + (1 - b_i) w_g \tag{18}$$

  with $w_d = \frac{\Sigma^{-1}\mu}{\mathbf{1}'\Sigma^{-1}\mu}$ and $w_g = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}$ from the two-fund separation result.
- Since all investors' portfolios are a combination of portfolios w_g and w_d, the market portfolio will be such a combination as well, i.e. we have $w_M = b_M w_d + (1 - b_M) w_g$.

The Zero-Beta CAPM (cont.)

- Given this, note that

$$\begin{aligned} Cov\left(\tilde{R}, \tilde{R_M}\right) &= \Sigma w_M = \Sigma\left[b_M w_d + (1-b_M) w_g\right] \\ &= b_M \frac{\Sigma\Sigma^{-1}\mu}{\mathbf{1}'\Sigma^{-1}\mu} + (1-b_M)\frac{\Sigma\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} \\ &= \frac{b_M}{B}\mu + \frac{1-b_M}{A}\mathbf{1} \end{aligned} \tag{19}$$

- Solving for μ implies the expected returns relation

$$\mu = \frac{B}{b_M}\left(\Sigma w_M - \frac{1-b_M}{A}\mathbf{1}\right) = \frac{B}{b_M}\left(Cov\left(\tilde{R}, \tilde{R_M}\right) - \frac{1-b_M}{A}\mathbf{1}\right) \tag{20}$$

- As in the CAPM case, our problem is to determine unknown quantities from market data (here b_M/B and $(1-b_M)/A$).

The Zero-Beta CAPM (cont.)

- Using the above relationship, for the market portfolio, we have

$$\sigma_M^2 = w_M' \Sigma w_M = \frac{b_M}{B} w_M' \mu + \frac{1-b_M}{A} w_M' \mathbf{1} = \frac{b_M}{B} \mu_M + \frac{1-b_M}{A} \quad (21)$$

- Now, take another minimum-variance portfolio w_z that is uncorrelated with the market portfolio. This yields

$$\sigma_{zM} = \frac{b_M}{B} \mu_z + \frac{1-b_M}{A} = 0 \quad (22)$$

- Solving these two equations, we have

$$\frac{b_M}{B} = \frac{\sigma_M^2}{\mu_M - \mu_z} \quad (23)$$

and

$$\frac{1-b_M}{A} = -\frac{b_M}{B} \mu_z = -\frac{\mu_z \sigma_M^2}{\mu_M - \mu_z} \quad (24)$$

The Zero-Beta CAPM (cont.)

- Inserting this into the expected returns relation $\mu = \frac{B}{b_M}\left(\Sigma w_M - \frac{1-b_M}{A}\mathbf{1}\right)$ yields

$$\begin{aligned}
\mu &= \frac{\mu_M - \mu_z}{\sigma_M^2}\left(\Sigma w_M + \frac{\mu_z \sigma_M^2}{\mu_M - \mu_z}\mathbf{1}\right) \\
&= \mu_z \mathbf{1} + (\mu_M - \mu_z)\frac{\Sigma w_M}{\sigma_M^2} \\
&= \mu_z \mathbf{1} + \frac{Cov\left(\tilde{R}, \tilde{R_M}\right)}{\sigma_M^2} \\
&= \mu_z \mathbf{1} + \beta(\mu_M - \mu_z) \qquad (25)
\end{aligned}$$

- This result is known as the zero-beta CAPM (or the Black CAPM or two-factor model). When there is no riskless asset, R is replaced with μ_z in the pricing equations, where μ_z is the expected return on a portfolio that is uncorrelated with the market portfolio.

The Zero-Beta CAPM (cont.)

- Note the difference between the usual CAPM and the zero-beta CAPM:
  - In the standard CAPM, all investors hold the market portfolio and therefore short sale restrictions are not binding.
  - In the zero-beta CAPM, investors may want to hold any portfolio on the mean-variance efficient set (the upper limb of the hyperbola). For these combinations to be attainable, short-selling of risky assets must be allowed.
- One can show that the expected return of the zero-beta portfolio is lower than the expected return of the global minimum-variance portfolio w_g, as follows.
- Recall that any minimum-variance portfolio can be formed as the combination of the market portfolio and the zero-beta portfolio. Since the two are uncorrelated,

$$\sigma_g^2 = \omega_z^2 \sigma_z^2 + (1 - \omega_z)^2 \sigma_M^2 \qquad (26)$$

The Zero-Beta CAPM (cont.)

- Minimizing this expression with respect to ω_z yields the first-order condition

$$\frac{d\sigma_g^2}{d\omega_z} = 2\omega_z\sigma_z^2 - 2(1-\omega_z)\,\sigma_M^2 = 0 \tag{27}$$

or, since both variances are positive,

$$0 < \omega_z = \frac{\sigma_M^2}{\sigma_M^2 + \sigma_z^2} < 1 \tag{28}$$

- Therefore , the expected return on the global minimum variance portfolio,

$$\mu_g = \omega_z\mu_z + (1-\omega_z)\,\mu_M \tag{29}$$

will lie between that on the market portfolio and that on the zero-beta portfolio, implying that the latter is on the lower limb of the hyperbola.

Outline

Shortcomings of CAPM

- The model assumes just two dates. In reality, people make multiperiod decisions, so they should have the opportunity to consume and rebalance portfolios repeatedly over time. For such extensions, see the intertemporal CAPM (ICAPM) of Robert Merton, and the consumption CAPM (CCAPM) of Douglas Breeden and Mark Rubinstein.
- Individuals have imperfect information and heterogeneous expectations.
- The specification of the market portfolio is difficult. In theory, the market portfolio should include all types of assets that are held by anyone - including works of art, real estate, human capital, etc. (Roll's critique). This brings a severe limitation with respect to empirical testing.

Shortcomings of CAPM (cont.)

- The model assumes that there are no taxes or transaction costs.
- Investors could have biased expectations. They can be, for example, overconfident. Also, some investors, like casino gamblers, will accept lower returns for higher risk.
- The model assumes that asset returns are jointly normally distributed.
- The model assumes that the variance of returns is an adequate measurement of risk.

Outline

Summary

- In equilibrium, the tangency portfolio becomes the market portfolio. The expected return of the market portfolio depends on the average risk aversion in the market.
- The intuition of the CAPM: expected return of any risky asset depends linearly on its exposure to the market risk, measured by β.
- Diversification is an important concept in finance. It builds on a powerful mathematical machine called Strong Law of Large Numbers.
- Beyond the CAPM:
  - Is β a good measure of risk exposure? What about the risk associated with negative skewness?
  - Could there be other risk factors?
  - Time varying risk aversion, and time varying β?

For Further Reading

1. Jagannathan and McGrattan, *The CAPM Debate*, 1995
   - a nice review of studies that support or challenge the CAPM.
2. Bernstein, Peter, *Capital Ideas: The Improbable Origins of Modern Wall Street*, 1992, Chapter 4.
   - about William Sharpe and his enormous single influence (he did however his work under the supervision of Harry Markowitz).
3. Burton, Jonathan, *Revisiting the CAPM*, 1998.
   - an eye-opening interview with William Sharpe.

Outline

- The Capital Market Line

$$\mu_e = R + \frac{\mu_t - R}{\sigma_t}\sigma_e \tag{30}$$

- The Security Market Line (CAPM relationship):

$$\mu_j - R = \beta_j\left(\mu_M - R\right), \quad \beta_j = \frac{Cov\left(\tilde{R_j}, \tilde{R_M}\right)}{Var\left(\tilde{R_M}\right)} = \rho_{jM}\frac{\sigma_j}{\sigma_M} \tag{31}$$

- The Zero-Beta CAPM

$$\mu = \mu_z \mathbf{1} + \beta\left(\mu_M - \mu_z\right) \tag{32}$$