Investments

Session 3. Asset Allocation (part II)

EPFL - Master in Financial Engineering
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Outline

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- Graphically, optimal portfolios are such that the investor's indifference curves are tangent to the mean-variance efficient set:

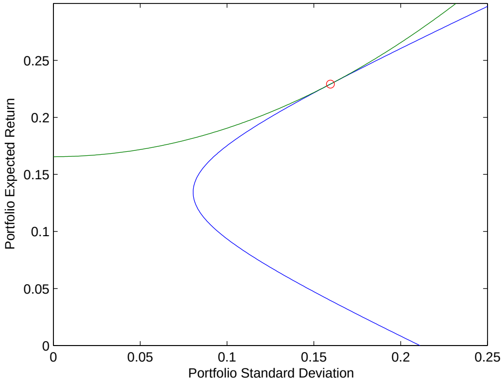


Figure 1: Optimal Portfolio with no Riskless Asset

The Portfolio Choice Problem in the Mean-Variance Case

- Consider a mean-variance investor with risk aversion a seeking to determine his optimal portfolio. His problem is

$$\max_{w}\left(w'\mu - \frac{a}{2}w'\Sigma w\right) \quad s.t. \quad w'\mathbf{1} = 1 \tag{1}$$

- Set up the Lagrangian

$$L = w'\mu - \frac{a}{2}w'\Sigma w + \lambda\left(1 - \mathbf{1}'w\right) \tag{2}$$

- The first order condition is

$$\frac{\partial L}{\partial w} = \mu - a\Sigma w - \lambda\mathbf{1} = 0 \tag{3}$$

The Portfolio Choice Problem in the Mean-Variance Case (cont.)

- This condition says that the marginal utility of investing in each asset is the same and equals λ.
- The investor's optimal portfolio as a function of λ is therefore

$$w = \frac{1}{a}\Sigma^{-1}(\mu - \lambda\mathbf{1}) \tag{4}$$

- To obtain λ, use the constraint $w'1 = 1$,

$$\frac{1}{a}\mathbf{1}'\Sigma^{-1}(\mu - \lambda\mathbf{1}) = 1 \tag{5}$$

solving,

$$\lambda = \frac{\mathbf{1}'\Sigma^{-1}\mu - a}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} = \frac{B-a}{A} \tag{6}$$

The Portfolio Choice Problem in the Mean-Variance Case (cont.)

- Inserting the solution for λ in the equation for w yields the composition of the investor's optimal portfolio as a function of the properties of the different assets and the investor's risk aversion,

$$w_o = \frac{1}{a}\Sigma^{-1}(\mu - \lambda \mathbf{1}) = \frac{1 - B/a}{A}\Sigma^{-1}\mathbf{1} + \frac{1}{a}\Sigma^{-1}\mu \tag{7}$$

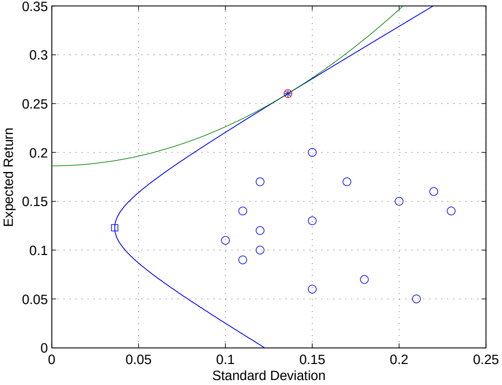
- The expected return and the variance of the optimal portfolio are

$$\mu_o = \mu' w_o = \frac{1 - B/a}{A}B + \frac{C}{a} \tag{8}$$

$$\sigma_o^2 = w_o' \Sigma w_o = \frac{1 - B/a}{A} + \frac{\mu_o}{a} \tag{9}$$

Example

Consider an economy with 15 risky assets, and an investor with mean-variance utility and a risk aversion coefficient of 8. Graphically, the optimal portfolio is: *[Matlab code to be shown in class]*



Outline

A New Efficient Frontier

- Suppose now that in addition to the N risky assets, a riskless asset with a rate of return of R is also available for investment.
- Then, the constraint $\mathbf{1}'w = 1$ disappears, as any excess or shortage of $\mathbf{1}'w$ compared to 1 can be offset by a riskless asset position. That is, the investment in the riskless asset is just $\omega_0 = 1 - \mathbf{1}'w$, where positive values denote lending and negative values borrowing (leverage).
- Taking the riskless asset position into account, the overall expected return on the portfolio is given by

$$\mu_p = \mu' w + \left(1 - \mathbf{1}' w\right) R \tag{10}$$

yielding the constraint

$$(\mu - R\mathbf{1})' w = \mu_p - R \tag{11}$$

A New Efficient Frontier (cont.)

- Therefore, the variance minimization problem with a riskless asset reads

$$\min_{w} \frac{1}{2} w' \Sigma w \quad s.t. \quad (\mu - R\mathbf{1})' w = \mu_p - R \tag{12}$$

- To solve this problem, set up the Lagrangian

$$L = \frac{w' \Sigma w}{2} + \gamma \left(\mu_p - R - (\mu - R\mathbf{1})' w \right) \tag{13}$$

- The first order conditions are

$$\begin{aligned} \frac{\partial L}{\partial w} &= \Sigma w - \gamma (\mu - R\mathbf{1}) = 0 \\ \frac{\partial L}{\partial \gamma} &= \mu_p - R - (\mu - R\mathbf{1})' w = 0 \end{aligned} \tag{14}$$

A New Efficient Frontier (cont.)

yielding the set of minimum variance portfolios

$$w = \gamma\Sigma^{-1}(\mu - R\mathbf{1}),\ \omega_0 = 1 - \mathbf{1}'w \tag{15}$$

- In order to determine γ, use the constraint $(\mu - R\mathbf{1})' w = \mu_p - R$:

$$(\mu - R\mathbf{1})' w = \gamma(\mu - R\mathbf{1})' \Sigma^{-1} (\mu - R\mathbf{1}) = \mu_p - R \tag{16}$$

yielding

$$\gamma = \frac{\mu_p - R}{(\mu - R\mathbf{1})' \Sigma^{-1} (\mu - R\mathbf{1})} = \frac{\mu_p - R}{C - 2RB + R^2 A} \tag{17}$$

with A, B, and C defined as before.

A New Efficient Frontier (cont.)

- The relationship between the expected return and variance of the minimum-variance portfolios when there is a riskless asset is given by:

$$\sigma_p^2(\mu_p) = w'\Sigma w = \gamma(\mu_p - R) = \frac{(\mu_p - R)^2}{C - 2RB + R^2 A} \tag{18}$$

- In standard deviation space, the minimum variance set is a pair of rays with intercepts R and slopes $\pm\sqrt{C - 2RB + R^2 A}$:

A New Efficient Frontier (cont.)

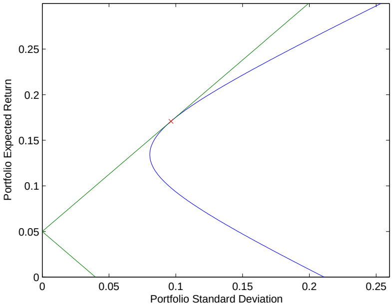


Figure 2: Minimum variance set with a riskless asset

Two Distinct Portfolios

- All minimum-variance portfolios are combinations of only two distinct portfolios ("mutual funds").
- Any two minimum-variance portfolios will span the set of all minimum-variance portfolios.
- Thus, a natural choice of portfolios is:
  1. The riskless asset, $w_R = 0$, $\omega_{0R} = 1$, and
  2. The tangency portfolio

$$w_t = \frac{\Sigma^{-1}(\mu - R\mathbf{1})}{B - AR}, \quad \omega_{0t} = 0 \tag{19}$$

- Let us consider the properties of the tangency portfolio. Note first that since $\omega_{0t} = 0$, the tangency portfolio is a member of the original risky-asset-only minimum-variance set.

Two Distinct Portfolios (cont.)

- Its expected return is given by

$$\mu_t = \mu' w_t = \frac{C - BR}{B - AR} \tag{20}$$

- Its variance is given by

$$\sigma_t^2 = w_t' \Sigma w_t = \frac{C - 2RB + R^2 A}{(B - AR)^2} \tag{21}$$

- Whenever $R < \mu_g = B/A$, tangency is on the upper limb of the hyperbola, and $\mu_t > \mu_g$ *[proof in class]*.

A New Optimal Portfolio

- Consider a mean-variance investor with risk aversion a seeking to determine his optimal portfolio.
- Recall that with a riskless asset, the portfolio's expected return is given by

$$\mu_p = R + w' \left(\mu - R\mathbf{1} \right) \tag{22}$$

- The expression for the variance is unchanged and reads

$$\sigma_p^2 = w' \Sigma w \tag{23}$$

- Hence, the investor's problem is

$$\max_w \left(\mu_p - \frac{a}{2} \sigma_p^2 \right) = \max_w \left(R + w' \left(\mu - R1 \right) - \frac{a}{2} w' \Sigma w \right) \tag{24}$$

A New Optimal Portfolio (cont.)

- The first-order condition is

$$\mu - R\mathbf{1} = a\Sigma w \tag{25}$$

yielding the optimal portfolio

$$w = \frac{1}{a}\Sigma^{-1}(\mu - R\mathbf{1}) \tag{26}$$

- Observe that this portfolio is proportional to the tangency portfolio

$$w_t = \frac{\Sigma^{-1}(\mu - R\mathbf{1})}{B - AR}, \quad \omega_{0t} = 0 \tag{27}$$

Graphically

- Graphically, the optimal portfolio is shown below:

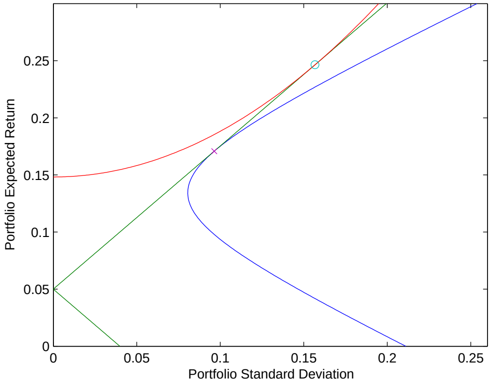


Figure 3: Optimal portfolio with a riskless asset

The Capital Market Line

- Hence, under homogeneous beliefs and mean-variance preferences, the following "separation" result holds
  - The optimal risky portfolio is identical for all investors, it is the tangency portfolio.
  - All investors hold a combination of the risk-free asset and the tangency portfolio, independently of preferences.
  - Individual preferences only determines the share of those two portfolios in the investor's overall investment. Investors with low risk-aversion make levered purchases of the tangency portfolio, investors with high risk-aversion invest part of their money in the risk-free asset, part in the tangency portfolio.
- The line that goes through the riskless asset and the tangency portfolio is the mean-variance efficient set and is called the Capital Market Line (CML).

The Capital Market Line (cont.)

- It gives the risk-return tradeoff available on the market. There is no better risk-return tradeoff than the one offered by the Capital Market Line.
- As a result, the portfolio selection process for all investors can be seen as consisting of two stages:
  1. Selection of the optimal risky portfolio (the tangency portfolio), and
  2. Based on individual risk tolerance, choice of the optimal combination between the riskless asset and the investment in the tangency portfolio.
- An interpretation of the tangency portfolio in terms of the risk aversion is instructive:
  - Suppose we have $i = 1, \ldots I$ investors and that investor i's risk aversion is a_i. Investor i's optimal portfolio is therefore

$$w_i = \frac{1}{a_i}\Sigma^{-1}(\mu - R\mathbf{1}) \tag{28}$$

The Capital Market Line (cont.)

- The tangency portfolio is given by

$$w_t = \frac{\Sigma^{-1}(\mu - R\mathbf{1})}{B - AR}, \quad \omega_{0t} = 0 \tag{29}$$

- Hence, we can view $B - AR$ as the "market's" risk aversion, a_M. Then, using $\Sigma^{-1}(\mu - R\mathbf{1}) = (B - AR)\, w_t = a_M w_t$ implies

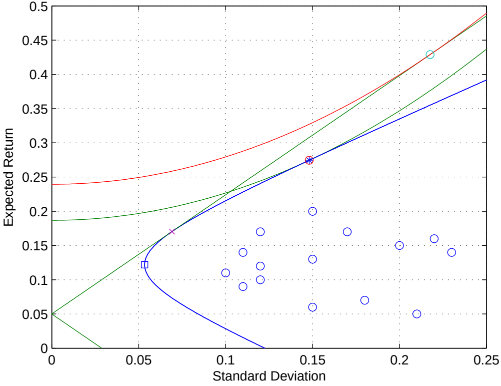
$$w_i = w_t \frac{B - AR}{a_i} = w_t \frac{a_M}{a_i} \tag{30}$$

Outline

Examples

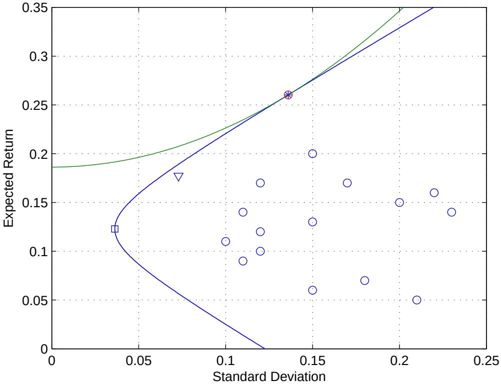
[Matlab codes for these examples will be shown in class]

- Take the previous example with 15 risky assets and add the risk-free asset with $R = 0.05$. The graph below shows the investor's utility gain:



Examples (cont.)

- Let's go back to the case without the risk-free asset, and assume that short sales are not allowed. We are forced to solve numerically for the new optimal portfolio. The solution is:



Examples (cont.)

Two more examples

- on the importance of correctly estimating expected return,
- on time diversification.

Outline

Summary

The Markowitz Portfolio Selection Model

1. The first step is to determine the risk-return opportunities available to the investor, summarized by the minimum-variance frontier. The part of the frontier that lies above the global minimum-variance portfolio is the efficient frontier.
2. The point of tangency between the efficient frontier and the Capital Market Line is the optimal risky portfolio, identical for all investors.
3. All investors hold a combination of the risk-free asset and the tangency portfolio, independently of preferences.
4. The mix of risky investments and risk-free investment vary with the degree of risk aversion of the investor

For Further Reading

1. Black, Fischer, "Estimating Expected Return". *Financial Analysts Journal*, 1993.
   - Estimating expected return is hard. Past average return is normally a highly innacurate estimate. Moreover, in finance we are interested in "ex ante" i.e., the future not "ex post" the past. The article outlines the problems of using theory or data to estimate expected return.
2. Brinson et al., "Determinants of Portfolio Performance", Financial Analysts Journal, 1986.
   - Much time is spent evaluationg individual portfolio managers or stocks - Is it worth it?
   - Brinson et al. measured the importance of asset allocation. Conclusion: in typical cases, variability in portfolio performance is driven by the asset allocation.
   - Similar conclusions in Ibbotson and Kaplan (2000).

For Further Reading (cont.)

3. Peter Bernstein, "Capital Ideas: The Improbable Origins of Modern Wall Street", 1992, Chapter 3.
   - About James Tobin and the logic of the Separation Theorem.
   - *The convenient fact that has just been proved is that the proportionate composition of the non-cash [i.e., risky] assets is independent of their aggregate share of the investment balance.*

Outline

- Optimal portfolios with no riskless assets

$$w_o = \frac{1 - B/a}{A} \Sigma^{-1} \mathbf{1} + \frac{1}{a} \Sigma^{-1} \mu \tag{31}$$

$$\mu_o = \frac{1 - B/a}{A} B + \frac{C}{a} \tag{32}$$

$$\sigma_o^2 = \frac{1 - B/a}{A} + \frac{\mu_o}{a} \tag{33}$$

- The new efficient set with a riskless asset

$$w = \gamma \Sigma^{-1} (\mu - R\mathbf{1}), \ \omega_0 = 1 - \mathbf{1}' w \tag{34}$$

$$\gamma = \frac{\mu_p - R}{C - 2RB + R^2 A} \tag{35}$$

$$\sigma_p^2(\mu_p) = \frac{(\mu_p - R)^2}{C - 2RB + R^2 A} \tag{36}$$

- The tangency portfolio

$$w_t = \frac{\Sigma^{-1}(\mu - R\mathbf{1})}{B - AR}, \quad \omega_{0t} = 0 \tag{37}$$

$$\mu_t = \frac{C - BR}{B - AR} \tag{38}$$

$$\sigma_t^2 = \frac{C - 2RB + R^2 A}{(B - AR)^2} \tag{39}$$

- Optimal portfolios with a riskless asset

$$w_o = \frac{1}{a}\Sigma^{-1}(\mu - R\mathbf{1}) \tag{40}$$