Investments

Session 2. Asset Allocation (part I)

EPFL - Master in Financial Engineering
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Outline

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The Portfolio Choice Problem

- Let us begin with a simple case: an individual has an initial wealth of W_0 and he must choose his current consumption C_0 and invest his savings $W_0 - C_0$ among N assets.
- Two constituent elements are necessary to formulate the problem:
  1. The time scale: We are dealing with a static situation characterized by two instants $t = 0$ and $t = 1$.
  2. The hypothesis regarding the attitude of individuals towards risk: we assume that the investor builds his portfolio so as to maximize his expected utility $E\left[U\left(C_0, \tilde{C}_1\right)\right]$.
- The investor's (random) consumption in period 1, $\tilde{C}_1$, will depend on how much he has saved in period 0, and on which assets he has invested in.

The Portfolio Choice Problem

- Let ω_n denote the fraction of wealth invested in the nth asset and $\tilde{R}_n$ the random return on the nth asset in period 1, i.e. the payoff of one unit of account invested in asset n in period 0.
- When formulating the investor's maximization problem, we must take two things into account:
  1. The size of the investor's savings in period 0, $W_0 - C_0$, constrains how much he can invest, and
  2. His portfolio holdings must sum to 1.
- In matrix notation, the formal statement of the portfolio problem is therefore

$$max_{C_0,w} E\left[U\left(C_0, (W_0 - C_0)\, w'\tilde{R}\right)\right] \quad s.t. \quad \mathbf{1}'w = 1 \qquad (1)$$

The Portfolio Choice Problem

- To solve this problem, form the Lagrangian

$$L = E\left[U\left(C_0, (W_0 - C_0)\, w'\tilde{R}\right)\right] + \lambda\left(1 - \mathbf{1}'w\right) \tag{2}$$

- The first order conditions are

$$\begin{aligned}
\frac{\partial L}{\partial C_0} &= E\left[U_1 - U_2 w'\tilde{R}\right] = 0 \\
\frac{\partial L}{\partial w} &= E\left[U_2 (W_0 - C_0)\,\tilde{R}\right] - \lambda \mathbf{1} = 0 \\
\frac{\partial L}{\partial \lambda} &= 1 - \mathbf{1}'w = 0
\end{aligned} \tag{3}$$

Interpretation

- Replace third FOC in the second to see that λ is the expected marginal utility of wealth:

$$\lambda = w' E\left[U_2\left(W_0 - C_0\right)\tilde{R}\right] \tag{4}$$

- Here is the economic interpretation of the portfolio solution:
  1. The first condition states that the investor consumes from wealth until the expected marginal utilities of consumption and savings are equal, $E\left[U_1\right] = E\left[U_2 w' \tilde{R}\right]$.
  2. The second condition says that the savings are allocated among the assets until each gives an equal contribution to expected marginal utility, $E\left[U_2\left(W_0 - C_0\right)\tilde{R}_n\right] = \lambda$.
- In static finance, one ignores the consumption part of the decision and only considers the investment decision aspect of the problem. Moreover, one considers utility over returns.

Interpretation

- Let W denote wealth available for consumption in period 1, $W = C_1 = (W_0 - C_0)\, w' R$. Then, the overall return on the portfolio, R_p, is given by $R_p = \frac{W}{W_0 - C_0} = w' R$.
- Writing $u(R_p) = U(C_0, W)$, we have

$$u'(R_p) = \frac{\partial u}{\partial R_p} = \frac{\partial U}{\partial W} \frac{dW}{dR_p} = U_2 (W_0 - C_0) \tag{5}$$

- Thus, we can rewrite the condition $E\left[U_2 (W_0 - C_0)\, \tilde{R}\right] - \lambda \mathbf{1} = 0$ as

$$E\left[u'\left(w' \tilde{R}\right) \tilde{R}\right] = E\left[u'\left(\tilde{R}_p\right) \tilde{R}\right] = \lambda \mathbf{1} \tag{6}$$

- Without some additional structure, the portfolio problem is hard to analyze. To make it tractable, restrictions are imposed on
  - the investor's utility function u, and/or
  - the distribution of asset returns, R.
- This brings us to the Mean-Variance Criterion.

Expected Utility Maximization and Mean-Variance

- The mean-variance criterion is widely used in practice. Under this criterion, a portfolio w can be optimal if it has minimum variance σ_p^2, given expected return μ_p.
- Formally, letting μ denote the vector of expected returns and Σ the variance-covariance matrix of returns, the portfolio's expected return is $\mu_p = \mu' w$ and its variance $\sigma_p^2 = w' \Sigma w$. Then, the minimum-variance portfolio with expected return μ_p is the solution to

$$min_w \frac{1}{2} w' \Sigma w \quad s.t. \quad \mu' w = \mu_p \tag{7}$$

- The main question is: under which conditions on (i) the distribution of asset returns and (ii) the investor's preferences is the mean-variance criterion consistent with expected utility maximization?

Expected Utility Maximization and Mean-Variance

- Answer: there are two particular cases
  1. The case of quadratic utility, which can be written as

$$u(R_p) = R_p - \frac{bR_p^2}{2} \tag{8}$$

     (computations for this case in class).
  2. The case of multivariate normally distributed asset returns and increasing and concave utility. The reason is that a normal distribution can be fully characterized by only two parameters: mean and variance. Thus, if the return on all assets is normally distributed, then the return on any portfolio of these assets will also be normally distributed, and mean and variance will be sufficient to describe it.

Proof of Equivalence in the Second Case

- Let $f\left(R_p;\mu,\sigma^2\right)$ denote the normal density of the portfolio return. Note that we drop p subscripts on μ and σ here. Then, expected utility is given by

$$V\left(\mu,\sigma^2\right) = E\left[u\left(\tilde{R}_p\right)\right] = \int_{-\infty}^{\infty} u(R_p)\, f\left(R_p;\mu,\sigma^2\right) dR_p$$
$$= \int_{-\infty}^{\infty} u(\mu+\sigma\varepsilon)\, n(\varepsilon)\, d\varepsilon \quad (9)$$

  where $\varepsilon = \frac{R_p-\mu}{\sigma}$ is standard normal with density $n(\varepsilon) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2}}$.

- Using Leibniz' rule, we see that nonsatiated investors will prefer higher expected returns:

$$\frac{\partial V}{\partial \mu} = \int_{-\infty}^{\infty} u'(\mu+\sigma\varepsilon)\, n(\varepsilon)\, d\varepsilon > 0 \quad (10)$$

Proof of Equivalence in the Second Case

- To derive the dependence of expected utility on portfolio variance, note that

$$\frac{\partial V}{\partial \sigma^2} = \frac{1}{2\sigma}\frac{\partial V}{\partial \sigma} = \frac{1}{2\sigma}\int_{-\infty}^{\infty} u'(\mu + \sigma\varepsilon)\,\varepsilon n(\varepsilon)\,d\varepsilon$$
$$= \frac{1}{2\sigma}\int_{-\infty}^{\infty} u''(\mu - \sigma\varepsilon)\,\sigma n(\varepsilon)\,d\varepsilon < 0 \quad (11)$$

- The last term is negative because u is concave. Thus, for normally distributed returns, the mean-variance criterion is consistent with expected utility maximization for all non-satiated risk-averse investors.

The Case of Exponential Utility

Example

(to be solved in class) Consider a risk-averse investor with exponential utility:

$$u(R_p) = -e^{-aR_p}, \quad a > 0 \tag{12}$$

If the distribution of returns is normal wiht mean μ and variance σ^2, then, maximizing expected utility is equivalent to maximizing $\mu - \frac{a}{2}\sigma^2$. Note that a is the investor's degree of risk aversion.

Interpretation

- Maximizing $\mu - \frac{a}{2}\sigma^2$ means that the investor is maximizing the portfolio's expected return minus a risk premium which is proportional to the variance of portfolio returns.
- Therefore, the investor's indifference curves are the solution to $\overline{V} = \mu - \frac{a}{2}\sigma^2$. Graphically, they are half-parabolas in mean-standard deviation space:

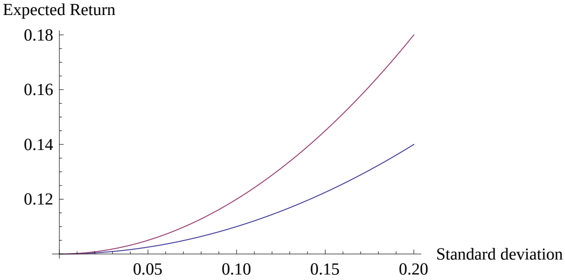
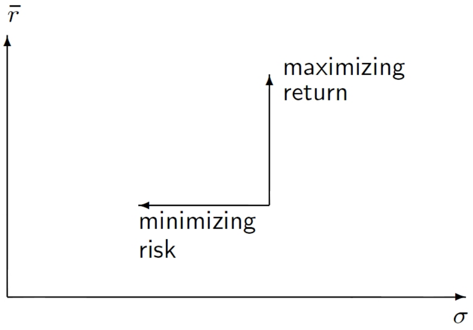


Figure 1: Indifference curves for $a = 2$ and $a = 5$.

Outline

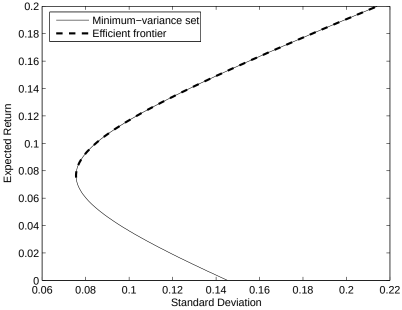
Efficient Portfolios

- The investor's problem is to choose a portfolio such that
  1. He minimizes the risk for a given expected return, and simultaneously,
  2. He maximizes the expected return for a given risk.
- We will call the set of the optimal portfolios mean-variance efficient.



Minimum-Variance vs Efficient Portfolios

- We will describe the properties of mean-variance efficient portfolios. To start with, we will consider the broader class of minimum-variance portfolios, i.e. the set that includes the single portfolio with the smallest variance at every level of expected return (it is easy to work with analytically).
- Graphically, the distinction between efficient and minimum-variance portfolios translates to the following:



The Mean-Variance Portfolio Problem

- We first consider the case without a riskless asset.
- Suppose there are N risky assets available for investment, and let μ denote the vector of asset expected returns and Σ the variance-covariance matrix of returns. Then, for any portfolio w, expected return is given by

$$\mu_p = w'\mu \tag{13}$$

  and portfolio variance equals

$$\sigma_p^2 = w'\Sigma w \tag{14}$$

- The minimum-variance portfolio with expected return μ_p is the solution $w(\mu_p)$ to

$$min_w \frac{1}{2} w'\Sigma w \quad s.t. \quad 1'w = 1, \mu' w = \mu_p \tag{15}$$

The Mean-Variance Portfolio Problem (cont.)

- To solve this problem, set up the Lagrangian

$$L = \frac{w'\Sigma w}{2} + \lambda\left(1 - 1'w\right) + \gamma\left(\mu_p - \mu' w\right) \tag{16}$$

- The first-order conditions are

$$\begin{aligned} \frac{\partial L}{\partial w} &= \Sigma w - \lambda 1 - \gamma\mu = 0 \\ \frac{\partial L}{\partial \lambda} &= 1 - 1'w = 0 \\ \frac{\partial L}{\partial \gamma} &= \mu_p - \mu' w = 0 \end{aligned} \tag{17}$$

- Hence, all minimum-variance portfolios are of the form

$$w = \lambda\Sigma^{-1}1 + \gamma\Sigma^{-1}\mu \tag{18}$$

The Mean-Variance Portfolio Problem (cont.)

- In order to determine the constants λ and γ, just use the two constraints and require that they be satisfied.
- For the first constraint, we have

$$1'w = 1'\left(\lambda\Sigma^{-1}1 + \gamma\Sigma^{-1}\mu\right) = 1 \tag{19}$$

or

$$1'\Sigma^{-1}1\lambda + 1'\Sigma^{-1}\mu\gamma = A\lambda + B\gamma = 1 \tag{20}$$

- For the second constraint, we have

$$\mu'w = \mu'\left(\lambda\Sigma^{-1}1 + \gamma\Sigma^{-1}\mu\right) = \mu_p \tag{21}$$

or

$$\mu'\Sigma^{-1}1\lambda + \mu'\Sigma^{-1}\mu\gamma = B\lambda + C\gamma = \mu_p \tag{22}$$

The Mean-Variance Portfolio Problem (cont.)

- Therefore, the composition of the minimum-variance portfolio with expected return μ_p is given by

$$\begin{aligned} w &= \lambda \Sigma^{-1} 1 + \gamma \Sigma^{-1} \mu \\ \lambda &= \frac{C - \mu_p B}{\Delta}, \quad \gamma = \frac{\mu_p A - B}{\Delta} \\ A &= 1' \Sigma^{-1} 1 > 0, \quad B = 1' \Sigma^{-1} \mu \\ C &= \mu' \Sigma^{-1} \mu > 0, \quad \Delta = AC - B^2 > 0 \end{aligned} \tag{23}$$

- Using the previous results, the relationship between expected return and variance on the minimum-variance set is given by

$$\begin{aligned} \sigma_p^2(\mu_p) &= w' \Sigma w = w' \Sigma \left(\lambda \Sigma^{-1} 1 + \gamma \Sigma^{-1} \mu \right) \\ &= \lambda w' 1 + \gamma w' \mu = \frac{A \mu_p^2 - 2 B \mu_p + C}{\Delta} \end{aligned} \tag{24}$$

Graphically

- In the mean-standard deviation space this is the equation of a hyperbola.
- It is important to note that the shape of the minimum-variance set depends not only on assets' expected returns and variances, but also on the correlation between their returns:

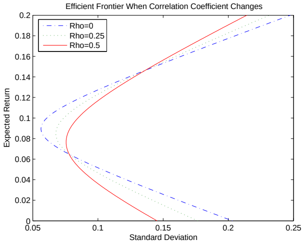


Figure 2: Minimum-variance set as a function of ρ

Properties and Graphical Description

- Let us analyze the properties of the minimum-variance set hyperbola
- The expected return on the global minimum variance portfolio w_g can be found by minimizing the expression for $\sigma_p^2(\mu_p)$:

$$\frac{d\sigma_p^2}{d\mu_p} = \frac{2A\mu_p - 2B}{\Delta} = 0 \tag{25}$$

yielding $\mu_g = B/A$.

- Inserting this value into the expression for σ_p^2 yields

$$\sigma_g^2 = \frac{1}{A} \tag{26}$$

Properties and Graphical Description (cont.)

- Using

$$\lambda = \frac{C - \mu_g B}{\Delta} = \frac{C - B^2/A}{\Delta} = \frac{1}{A} \tag{27}$$

and

$$\gamma = \frac{\mu_g A - B}{\Delta} = 0 \tag{28}$$

the composition of the global minimum-variance portfolio w_g is given by

$$w_g = \lambda \Sigma^{-1} 1 = \frac{\Sigma^{-1} 1}{A} = \frac{\Sigma^{-1} 1}{1 \Sigma^{-1} 1} \tag{29}$$

- Let us now determine the slope of the asymptotes of the hyperbola. First, compute the slope of the hyperbola at any point,

$$\frac{d\mu_p}{d\sigma_p} = \frac{d\mu_p}{d\sigma_p^2}\frac{d\sigma_p^2}{d\sigma_p} = \frac{\Delta}{2A\mu_p - 2B} 2\sigma_p = \frac{\Delta}{A\mu_p - B}\sqrt{\frac{A\mu_p^2 - 2B\mu_p + C}{\Delta}} \tag{30}$$

Properties and Graphical Description (cont.)

- Taking the limit of this expression as $\mu_p \to \pm\infty$ then yields

$$\lim_{\mu_p \to \pm\infty} \frac{d\mu_p}{d\sigma_p} = \pm\sqrt{\frac{\Delta}{A}} \tag{31}$$

- This completes our description of the hyperbola.
- Note that the covariance of any asset or portfolio with the global minimum variance portfolio w_g is $1/A$:

$$Cov\left(\tilde{R}_g, \tilde{R}_p\right) = w_g' \Sigma w_p = \frac{1'\Sigma^{-1}}{A} \Sigma w_p = \frac{1' w_p}{A} = \frac{1}{A} \tag{32}$$

- All minimum-variance portfolios can be seen as portfolio combinations of only two distinct portfolios. This result is called *two-fund separation* *[to be shown in class]*

Systematic vs Idiosyncratic Risk

- So far, we have been concerned with the properties of minimum-variance portfolos. given μ and Σ, we saw that we can fully describe the mean-variance efficient set.
- We mentioned earlier that correlation between asset returns is a key driver of the shape of the efficient set. Low or negative correlation seemed to lead to a more favorable efficient set in terms of the available risk-return menu.

Example

(to be solved in class). Let us analyze the issue of diversification in more detail. To do so, consider an equally-weighted portfolio. This portfolio will not be efficient in most cases, but let us analyze its properties to get some intuition for what is actually happening. The weight of each of the N assets in an equally-weighted portfolio is $\omega_n = 1/N$. The figure below shows that most of the benefits of diversification arise with 20 to 30 assets.

Systematic vs Idiosyncratic Risk (cont.)

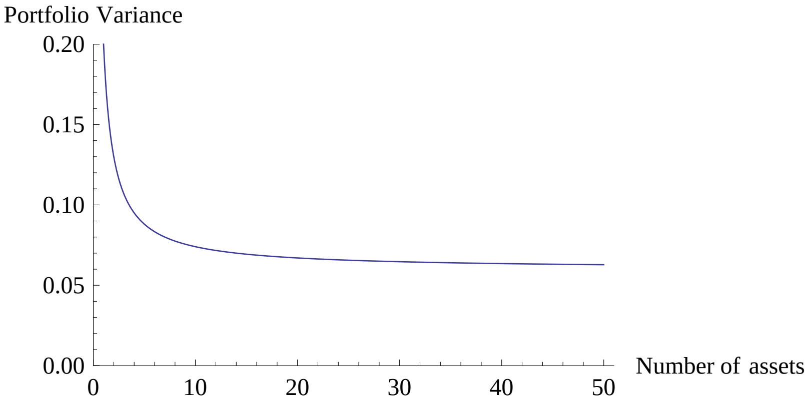


Figure 3: Diversification

- Therefore, one can distinguish two kinds of risk:

Systematic vs Idiosyncratic Risk (cont.)

1. idiosyncratic risk is specific to a given asset and can be diversified,
2. systematic risk arises from the correlation/covariance in asset returns and cannot be diversified

- A very important result of asset pricing theory is that the market pays no risk premium for bearing idiosyncratic risk because it can be avoided by diversification. We will return to this point later in the course.

Outline

Summary

- If returns are log-normal and utility is exponential $\Rightarrow$ mean-variance framework.
- The investment set:
  - is a hyperbola with focal point the global minimum-variance portfolio.
  - depends on $\mu$and Σ and implicitely on ρ.
- Two-fund separation: all minimum-variance portfolios can be obtained as a combination of the global minimum-variance portfolio w_g and the portfolio w_d. Soon we will see why this result is so powerful.
- Market pays no risk premium for bearing idiosyncratic risk.

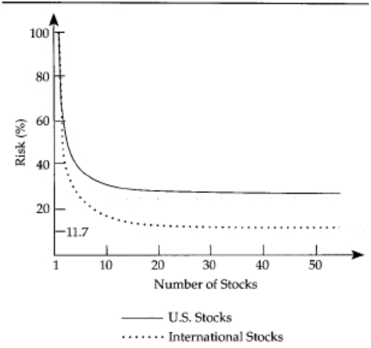
For Further Reading

1. Peter Bernstein, "Capital Ideas: The Improbable Origins of Modern Wall Street", 1992, Chapter 2.
   - About Harry Markowitz, the Nobel Prize, etc.
   - *I was struck with the notion that you should be interested in risk as well as return.*
2. Elton, Edwin, and Gruber, Martin, "The Rationality of Asset Allocation Recommendations". *The Journal of Financial and Quantitative Analysis*, 2000.
   - Helps to better understand the efficient set mathematics.
   - The recommendations on asset allocation presented by the investment advisors are consistent with modern portfolio theory.

For Further Reading (cont.)

3. Solnik, Bruno, "Why Not Diversify Internationally Rather Than Domestically?", Financial Analysts Journal, 1974.
   - Substantial advantages in risk reduction can be attained through portfolio diversification in foreign securities.

Figure 9. International Diversification



Outline

The composition of the minimum-variance portfolio

$$\begin{aligned} w &= \lambda\Sigma^{-1}1+\gamma\Sigma^{-1}\mu \\ \lambda &= \frac{C-\mu_p B}{\Delta}, \quad \gamma=\frac{\mu_p A-B}{\Delta} \\ A &= 1'\Sigma^{-1}1>0, \quad B=1'\Sigma^{-1}\mu \\ C &= \mu'\Sigma^{-1}\mu>0, \quad \Delta=AC-B^2>0 \end{aligned}$$

Relation between expected return and variance for the minimum variance set

$$\sigma_p^2(\mu_p)=\frac{A\mu_p^2-2B\mu_p+C}{\Delta}$$

The expected return, variance and composition of the global minimum-variance portfolio

$$\mu_g=B/A, \quad \sigma_g^2=1/A, \quad w_g=\frac{\Sigma^{-1}1}{1\Sigma^{-1}1}$$